Chapter 7. Integrals and Transcendental Functions

7.4. Relative Rates of Growth

Note. On page 444, the text describes how fast the exponential function $f(x) = e^x$ grows. If I graph $y = e^x$ on the whiteboard with the axes calibrated by centimeters. We get the following values:

x (cm)	$\approx e^x$ (cm)	Approximate Distance
0	1	width of marker
1	2.72	1 inch
2	7.4 (3 in)	Diameter of a Baseball
3	20 (8 in)	Diameter of a Cantelope
4	55 (22 in)	Top of Whiteboard
5	148 (5 ft)	
6	403 (13 ft)	Past the Ceiling
7	1096 (37 ft)	
8	2980 (99 ft)	
9	8103 (270 ft)	Football Field
10	22,026 (734 ft)	
12	162,755 (1 mile)	
15	3,269,017 (20 miles)	
17	24,154,953 (150 miles)	Low Earth Orbit
24	2.65×10^{10} (164,596 miles)	2/3 to Moon
30	1.07×10^{13} (66,402,674 miles)	2/3 to Sun
43	4.73×10^{18} (5 light-years)	Nearest Star to Solar System
56	2.09×10^{24} (2 million light-years)	Andromeda Galaxy
65	1.68×10^{28} (15 billion light-years)	Edge of the Universe

In contrast to this, is the slow rate of growth of the logarithmic function $\ln x$.

Definition. Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f *grows faster than* g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \text{ or equivalently if } \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

2. f and g *grow at the same rate* as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

Example. Let $f(x) = e^x$ and $g(x) = x^n$ for some positive integer n. Show that f grows faster than g.

Definition. A function f is *of smaller order than* g as $x \to \infty$ if $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, We indicate this by writing $f = o(g)$ ("f is little-oh of g").

Definition. Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is *of at most the order of* g as $x \to \infty$ if there is a positive integer M for which $\dfrac{f(x)}{g(x)} \leq M$ for x sufficiently large. We indicate this be writing $f = O(g)$ ("f is big-oh of g).

Note. If $f = O(g)$, then f and g are asymptotically multiples of each other.

Example. Page 449 10e, 11 and 18.