Chapter 8. Techniques of Integration

8.4 Integration of Rational Functions by Partial Fractions

Note. Consider
$$\int \frac{1}{1-x^2}\,dx.$$
We can see that we have the algebraic identity
$$\frac{1}{1-x^2} = \frac{1/2}{1+x} + \frac{1/2}{1-x}.$$
Then we have
$$\begin{aligned}
\int \frac{1}{1-x^2}\,dx &= \int \frac{1/2}{1+x}\,dx + \int \frac{1/2}{1-x}\,dx \\
&= \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| + C \\
&= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C \\
&= \frac{1}{2}\ln\frac{1+x}{1-x} + C \text{ if } |x|<1 \\
&= \tanh^{-1}x + C \text{ if } |x|<1
\end{aligned}$$

What simplified this computation was breaking up the denominator and "undoing" the common denominator process. This will be the idea of the method of this section, which is called the method of partial fractions.

Note. A polynomial with real coefficients can be factored into linear factors $(x - r_i)$ and irreducible quadratics $(x^2 + p_j x + q_j)$. To show this, we need to know some complex variables (this result is presented in our Complex Variables class, MATH 4337/5337).

Note. Method of Partial Fractions. If f and g are polynomials, to integrate f/g:

1. If the degree of f is greater than or equal to the degree of g, perform long division.

2. Factor g into linear factors $(x - r)$ and irreducible quadratics $(x^2 + px + q)$.

3. For each linear factor $(x - r)$ of g of order m (that is, $(x - r)$ divides g m times), take the partial fractions

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

4. For each irreducible quadratic factor $x^2 + px + q$ of g of order n, take the partial fractions

$$\frac{B_1 x + C_1}{(x^2+px+q)} + \frac{B_2 x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_n x + C_n}{(x^2+px+q)^n}.$$

5. Set f/g equal to the sum of all partial fractions.

6. Evaluate the A's, B's, and C's and integrate with the methods you already know.

Examples. Page 479 numbers 16, 20, and 26, page 580 number 42.

Note. The following is the "Heaviside Cover-Up Method."

Example. Page 477 Example 7.