Chapter 8. Techniques of Integration

8.6 Numerical Integration

Note. If we start with a regular partition, then we can approximate definite integrals using trapezoids instead of rectangles.

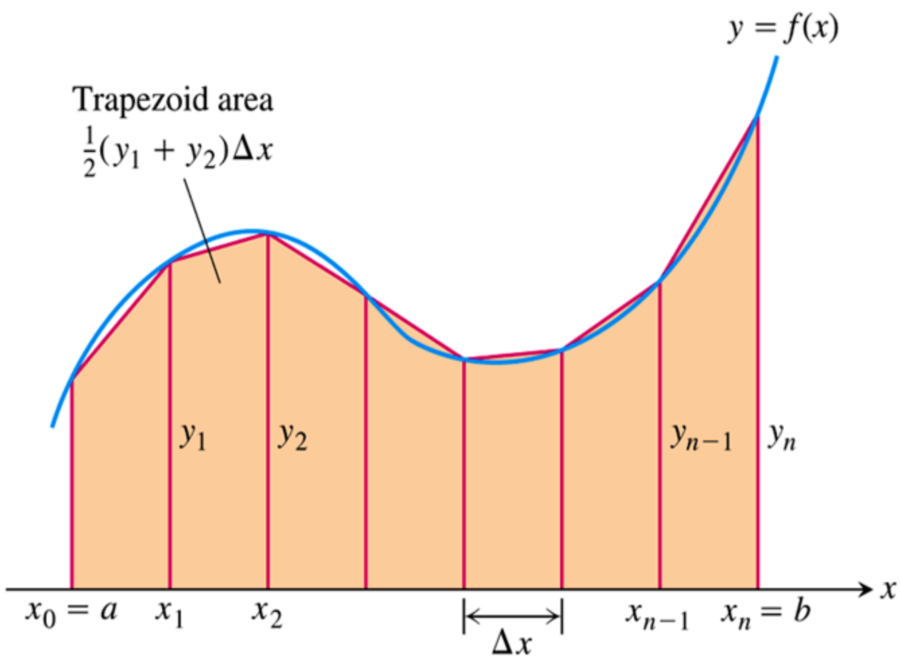

Figure 8.7 page 487

We let $\Delta x = \dfrac{b-a}{n}$ and the area of the kth trapezoid is

$$(\text{base}) \times (\text{average height}) = \frac{y_{k-1} + y_k}{2}\Delta x = \frac{1}{2}(y_{k-1} + y_k)\Delta x.$$

So our estimate is

$$T = \sum_{k=1}^{n} \frac{1}{2}(y_{k-1} + y_k)\Delta x = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).$$

Definition. In the *Trapezoid Rule*, the integral $\displaystyle\int_a^b f(x)\,dx$, is approximated by

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

This approximation is based on a regular partition of $[a, b]$ where $\Delta x_k = (b-a)/n$, $x_k = a + k\Delta x$, and $y_k = f(x_k)$.

Theorem 1a. We can estimate the error involved in using the Trapezoid Rule to approximate a definite integral. If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then

$$|E_T| = \left|\int_a^b f(x)\,dx - T\right| \le \frac{b-a}{12}h^2 M = \frac{M(b-a)^3}{12n^2}$$

where $h = \Delta x = (b-a)/n$.

Note. If $f(x) = mx + b$ then $f''(x) \equiv 0$ and $E_T = 0$. So the Trapezoid Rule gives exact values for such functions.

Examples. Page 494 number 8 I abc.

Note. Instead of approximating $y = f(x)$ with straight line segments, we can approximate it with parabolas. We then integrate to find the area under the parabolas. This leads to Simpson's Rule.

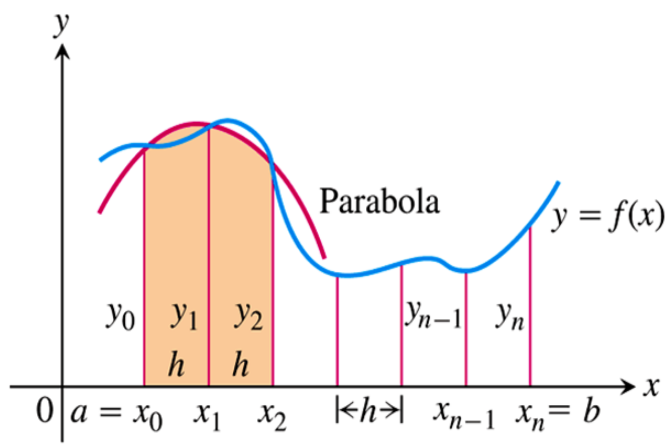

Figure 8.9 page 488

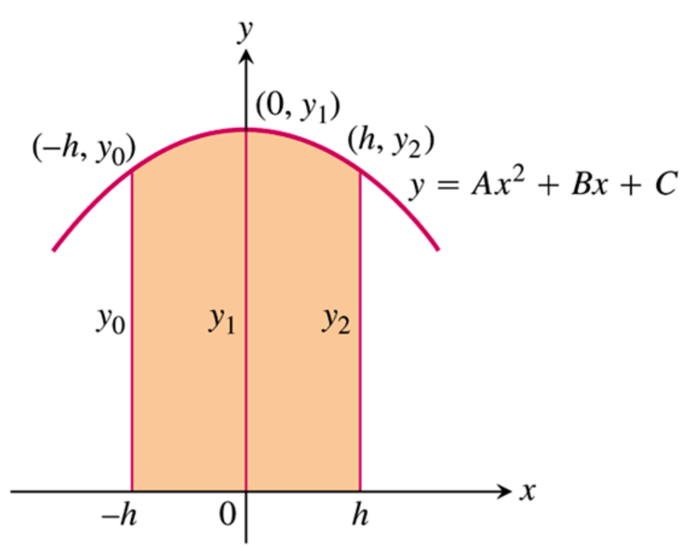

Figure 8.10 page 489. The shaded area is $\frac{h}{3}(y_0 + 4y_1 + y_2)$.

Definition. In *Simpson's Rule*, the integral $\int_a^b f(x)\,dx$, is approximated by

$$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

This approximation is based on a regular partition of $[a,b]$ of size n where n is even, and where $\Delta x = h = (b-a)/n$, $x_k = a + kh$, and $y_k = f(x_k)$.

Theorem 1b. We can estimate the error involved in using Simpson's Rule to approximate a definite integral. If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a,b]$, then

$$|E_S| = \left| \int_a^b f(x)\,dx - S \right| \le \frac{b-a}{180} h^4 M = \frac{M(b-a)^5}{180n^4}$$

where $h = (b-a)/n$.

Note. If f is a third degree polynomial then $f^{(4)}(x) \equiv 0$ and $E_S = 0$. So Simpson's Rule gives exact values for such functions.

Examples. Page 494 number 8 II abc, and page 494 number 24.