Chapter 8. Techniques of Integration

8.7 Improper Integrals

Note. In this section, we are interested in finding the area under a curve over an infinite interval. This arises, in particular, in probability and statistics when looking at, for example, the area under the normal distribution.

Definition. Integrals with infinite limits of integration are *improper integrals of Type I*:

1. If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty,b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

In parts 1 and 2, if the limit is finite, the improper integral *converges* and the limit is the *value* of the improper integral. If the limit fails to exist, the improper integral *diverges*. In part 3, the integral on the left-hand side of the equation *converges* if both improper integrals on the right-hand side converge; otherwise it *diverges* and has no value.

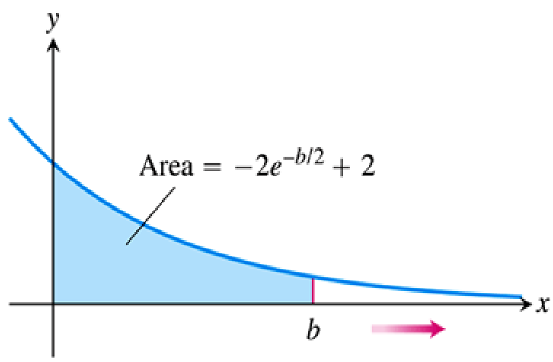


Figure 8.13b page 496

Example. Example 2 page 497: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Example. Evaluate $\displaystyle\int_{1}^{\infty} \frac{1}{x}\,dx$.

Definition. Integrals of functions that become infinite at a point within the interval of integration are *improper integrals of Type II*:

1. If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

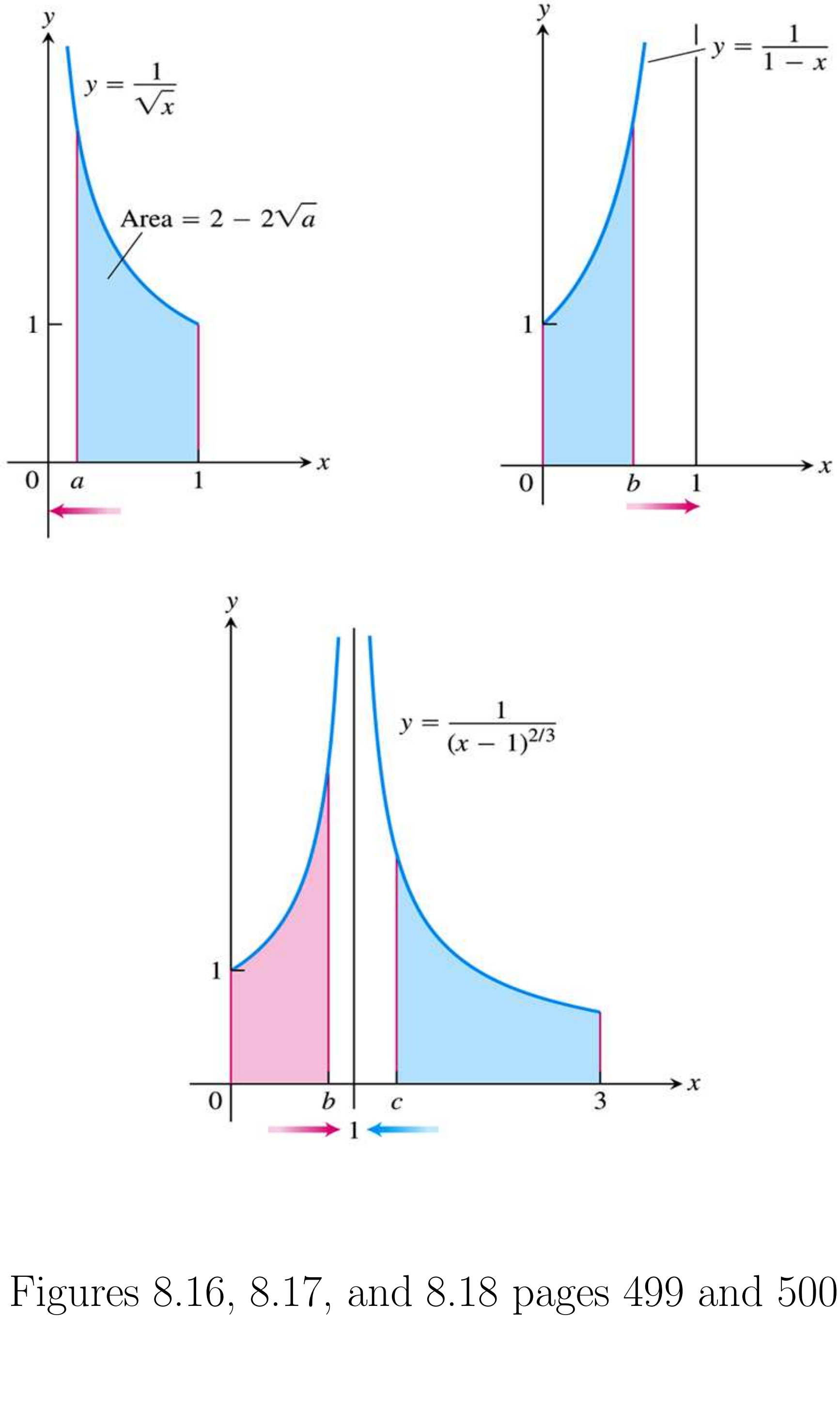
2. If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is continuous on $[a, c) \bigcup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In parts 1 and 2, if the limit is finite, the improper integral *converges* and the limit is the *value* of the improper integral. In the limit fails to exist, the improper integral *diverges*. In part 3, the integral on the left-hand side of the equation *converges* if both integrals on the right-hand side values; otherwise it *diverges*.



Figures 8.16, 8.17, and 8.18 pages 499 and 500.

Example. Page 505 number 28.

Note. When we cannot evaluate an improper integral directly (often the case in practice) we first try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can then use numerical methods to approximate its value.

Theorem 2. Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges.

2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

Example. Page 505 Number 52.

Theorem 3. Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

Example. Page 505 Number 48.

Note. Sometimes the Limit Comparison Test is easier than the Direct Comparison Test, since we don't have to worry about inequalities.

Example. Evaluate $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2+1}}$.

Example. Page 506 number 74 (Gabriel's Horn).

Note. Consider the function $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$. This is the "bell curve" or *normal distribution*. We are interested in showing that $\displaystyle\int_{-\infty}^\infty f(x)\,dx = 1$ (since this is a probability distribution). This is equivalent to evaluating $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx$. Unfortunately, it *IS IMPOSSIBLE* to antidifferentiate e^{-x^2}...well, not so fast! Maybe we can try one more trick...This story is to be concluded in section 10.7.