

Chapter 9. First-Order Differential Equations

9.1 Solutions, Slope Fields, and Euler's Method

Definition. A *first-order differential equation* is a relation

$$\frac{d}{dx}[y(x)] = f(x, y(x))$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy -plane. A *solution* of this equation is a differentiable function $y = y(x)$ defined on an interval of x -values such that

$$\frac{d}{dx}[y(x)] = f(x, y(x))$$

on that interval. The *general solution* is the set of all solutions and involves a constant of integration. A *particular solution* is a solution satisfying a given initial condition $y(x_0) = y_0$. The first-order differential equation together with an initial condition form a *first-order initial value problem*.

Note. We encountered this idea in Section 7.2.

Note. We can graph little hash-marks to indicate the slope $dy/dx = f(x, y)$ at various points in the xy -plane to give some idea of the “flow” of a solution. Such a collection of hash-marks is called a *slope field* for the differential equation.

Example. The slope field for the differential equation $\frac{dy}{dx} = y - x$ is given below, along with the particular solution which passes through the point $(0, 2/3)$

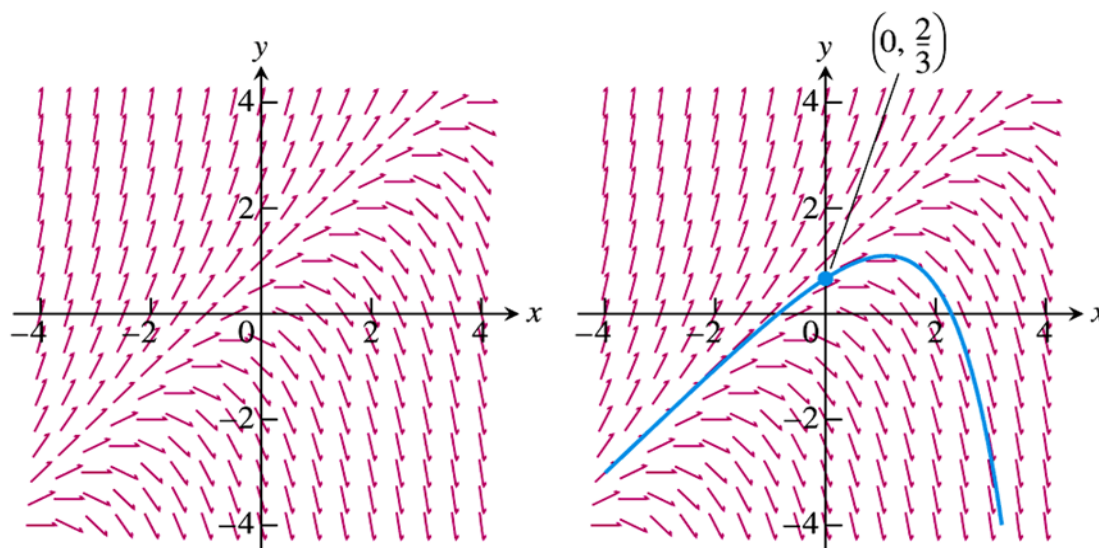


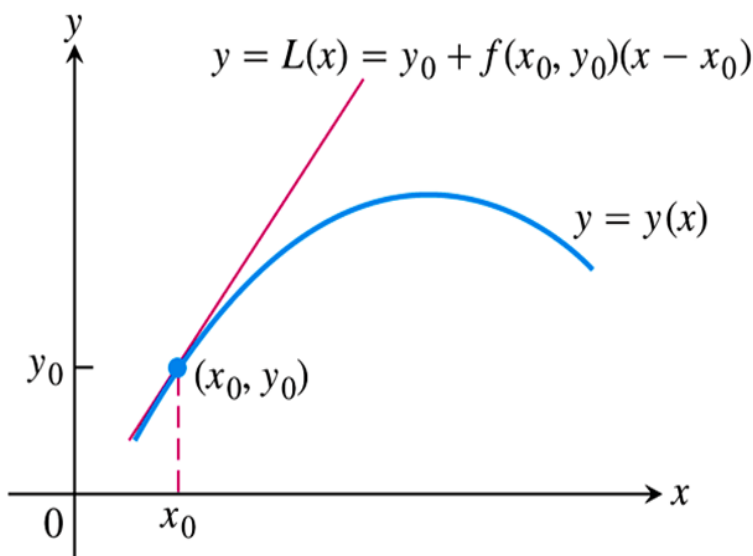
Figure 9.2 page 516

Example. Page 520 number 4.

Note. (Euler's Method) Given a differential equation $dy/dx = f(x, y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution $y = y(x)$ by its linearization

$$L(x) = y(x_0) + y'(x_0)(x - x_0) \text{ or } L(x) = y_0 + f(x_0, y_0)(x - x_0).$$

The function $L(x)$ gives a good approximation to the solution $y(x)$ in a short interval about x_0 :



Page 517 Figure 9.4

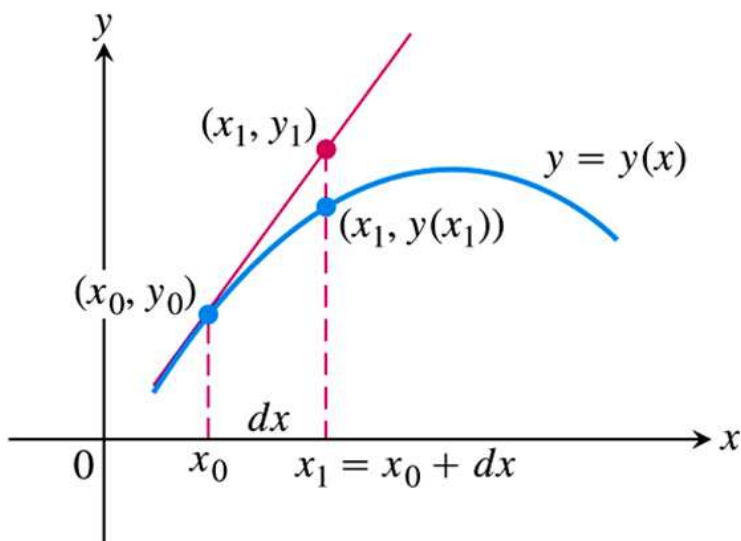
The basis of Euler's Method is to patch together a string of linearizations to approximate the curve over a long stretch.

We know the point (x_0, y_0) lies on the solution curve. Suppose that we specify a new value for the independent variable to be $x_1 = x_0 + dx$. If

the increment dx is small, then

$$y_1 = L(x_1) = y_0 + f(x_0, y_0)dx$$

is a good approximation to the exact solution value $y = y(x_1)$. So from the point (x_0, y_0) , which lies exactly on the solution curve, we have obtained the point (x_1, y_1) which lies very close to the point $(x_1, y(x_1))$ on the solution curve:



Page 517 Figure 9.5

Using the point (x_1, y_1) and the slope $f(x_1, y_1)$ of the solution curve through (x_1, y_1) we take a second step. Setting $x_2 = x_1 + dx$, we use the linearization of the solution curve through (x_1, y_1) :

$$y_2 = y_1 + f(x_1, y_1)dx.$$

This gives the next approximation (x_2, y_2) to values along the solution curve $y = y(x)$ with slope $f(x_2, y_2)$ to obtain the third approximation

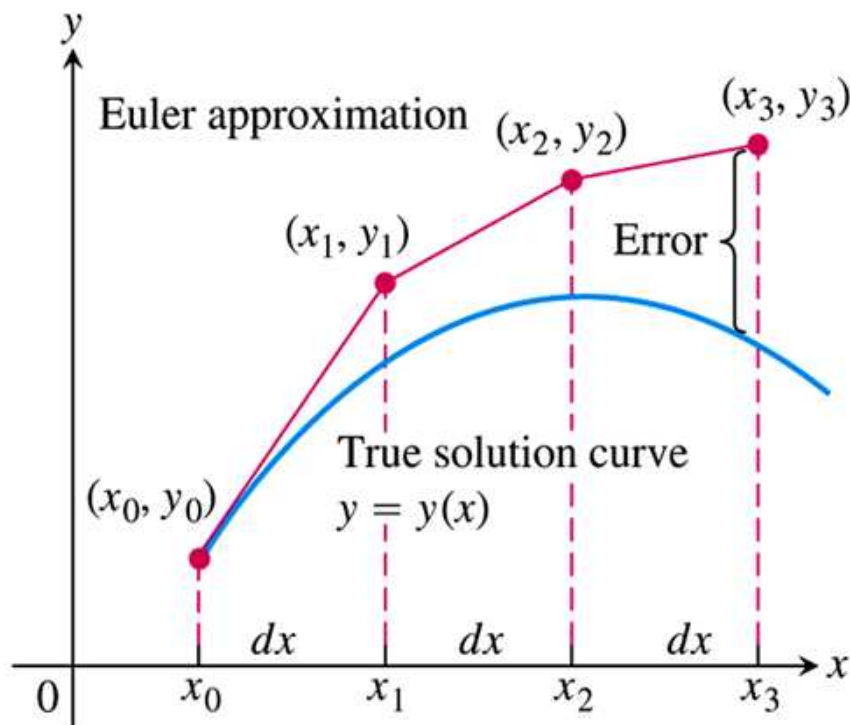
$$y_3 = y_2 + f(x_2, y_2)dx,$$

and so on. In general, we have

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})dx.$$

We are literally building an approximation to one of the solutions by following the direction of the “slope field” of the differential equation.

Note. We have $x_n = x_0 + n dx$.



Example. Page 521 number 12.

Note. It might be tempting to reduce the step size more and more in Euler's Method in order to obtain better accuracy. However, this requires additional computation time and more importantly adds to the buildup of round-off errors due to the approximate representations of numbers. The analysis of errors and the investigation of methods to reduce it when making numerical calculations are important and are explored in a Numerical Analysis class. In fact, in such a class you will see that there are numerical methods more accurate and more sophisticated than Euler's method.

Example. Page 663 Example 3. Use Euler's Method to solve the I.V.P.

$$y' = 1 + y, \quad y(0) = 1,$$

on the interval $x \in [0, 1]$, starting at $x_0 = 0$ and taking $dx = 0.1$. Compare the approximations with the values of the exact solution $y = 2e^x - 1$. (See Table 9.2 on page 520.)