Chapter 9. First-Order Differential Equations

9.3 Applications

Note. In mechanics, it is common to assume that the force of resistance of a moving object is proportional to the velocity of the object. With velocity as v, mass as m and time as t, we have from Newton's law of motion,

$$\text{Force} = \text{mass} \times \text{acceleration}$$

or

$$m\frac{dv}{dt} = -kv \text{ or } \frac{dv}{dt} = -\frac{k}{m}v$$

where $k > 0$. As with all the above problems, we integrate to find that

$$v = v_0 e^{-(k/m)t}.$$

Example. Page 533 Number 2.

Note. The exponential growth model for population growth assumes infinite resources and unlimited growth that results from the growth rate constant. It is more realistic to assume that the environment has a *carrying capacity* M that represents the maximum population size which the environment can sustain in the long run. This leads us to the *logistic growth model.*

Definition. The differential equation

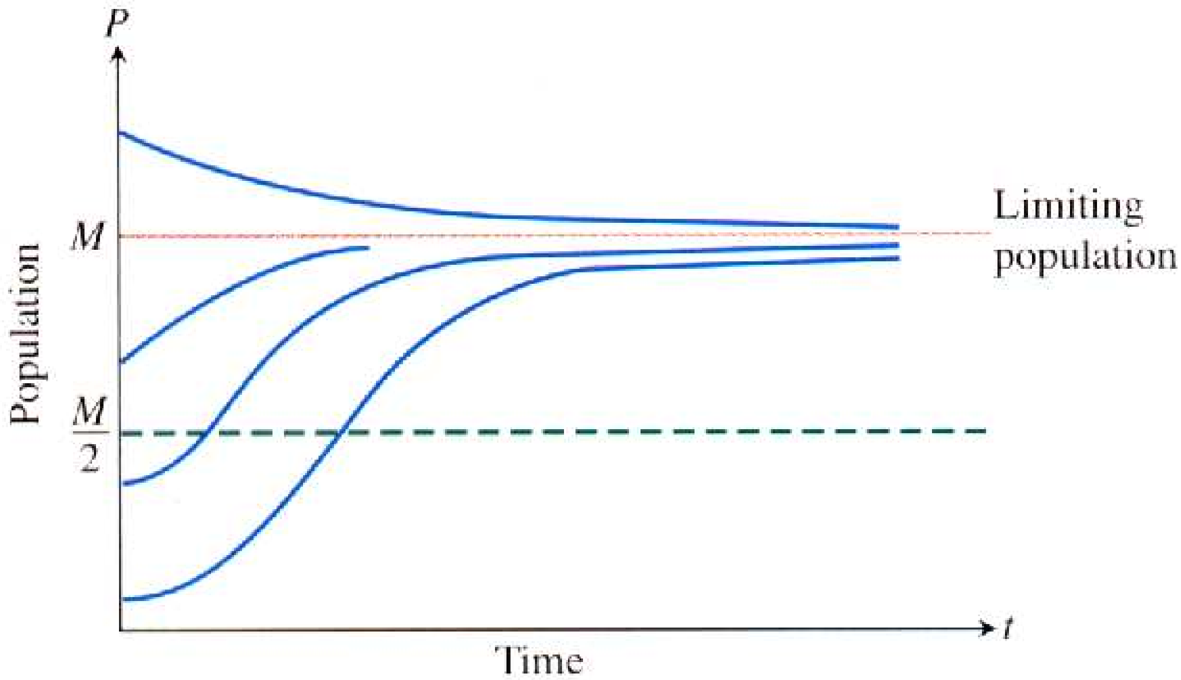
$$\frac{dP}{dt} = r(M - P)P$$

is the *logistic growth model.* The parameter r is the *logistic growth constant* and M is the carrying capacity.

Example. Show that the general solution to the logistic equation is

$$P = \frac{M}{1 + Ae^{-rMt}}$$

where A is some constant (to be determined from initial population size). The graph of this function is



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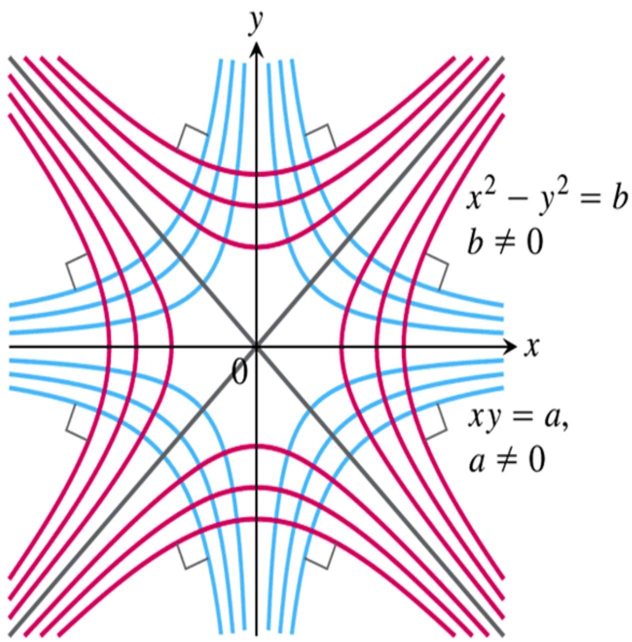
Note. An *orthogonal trajectory* of a family of curves is a curve that intersects each curve of the family at right angles, or *orthogonally*.



Page 531 Figure 9.13

Example. Page 534 Number 6.

Note. A chemical fluid (or gas) runs into a container containing a different fluid (or gas). The solution is then mixed so that the concentration of each fluid is the same throughout. At the same time, some of the fluid is removed from the container. The differential equation describing the process is based on the formula:

$$\begin{array}{c}\text{Rate of Change}\\ \text{of amount}\\ \text{in container}\end{array} = \left(\begin{array}{c}\text{rate at which}\\ \text{chemical arrives}\end{array}\right) - \left(\begin{array}{c}\text{rate at which}\\ \text{chemical departs.}\end{array}\right)$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure of the chemical at time t is

$$\begin{aligned}\text{Departure rate} &= \left(\begin{array}{c}\text{concentration in}\\ \text{container at time } t\end{array}\right)(\text{outflow rate})\\ &= \frac{y(t)}{V(t)}(\text{outflow rate}).\end{aligned}$$

Therefore the relevant equation is

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)}(\text{outflow rate}).$$

Example. Page 534 number 14.