Chapter 9. First-Order Differential Equations

9.5 Systems of Equations and Phase Planes

Definition. A general system of two first-order differential equations may take the form

$$\begin{aligned}\frac{dx}{dt} &= F(x,y),\\ \frac{dy}{dt} &= G(x,y).\end{aligned}$$

Such a system of equations is called *autonomous* because dx/dt and dy/dt do not depend on the independent variable t, but only on the dependent variables x and y. A *solution* of such a system consists of a pair of functions $x(t)$ and $y(t)$ that satisfies both of the differential equations simultaneously for every t in some interval.

Note. To gain insight into the solutions, we look at both dependent variables together by plotting the points $(x(t), y(t))$ in the xy-plane starting at some specified point. Therefore the solution functions define a solution curve through the specified point, called a *trajectory* of the system. The xy-plane itself, in which these trajectories reside, is referred to as the *phase plane*.

Example. A Competitive-Hunter Model.

Imagine two species of fish, say trout and bass, competing for the same limited resources (such as food and oxygen) in a certain pond. We let $x(t)$ represent the number of trout and $y(t)$ the number of bass living in the pond at time t.

As time passes, each species breeds, so we assume its population increases proportionally to its size. Taken by itself, this would lead to exponential growth in each of the two populations. The two species are in competition. A large number of bass tends to cause a decrease in the number of trout, and vice-versa. Our model takes the size of this effect to be proportional to the frequency with which the two species interact, which in turn in proportional to xy. The model describing this is:

$$\begin{aligned} \frac{dx}{dt} &= (a - by)x, \\ \frac{dy}{dt} &= (m - nx)y. \end{aligned}$$

This is the *competitive-hunter model.*

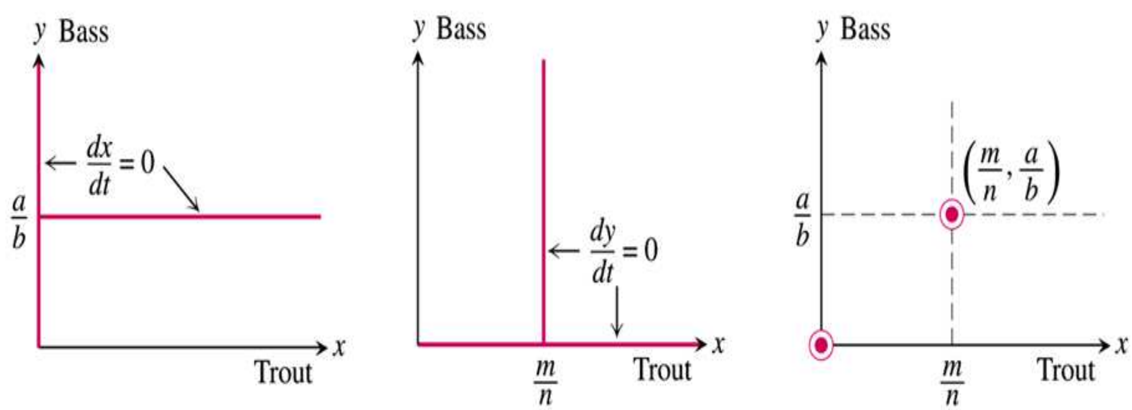
The only part of the phase plane which is of interest is the first quadrant, since neither $x < 0$ nor $y < 0$ are physically meaningful. The populations are constant where $dx/dt = 0$ and where $dy/dt = 0$. This occurs when:

$$(a - by)x = 0,$$

$$(m - nx)y = 0.$$

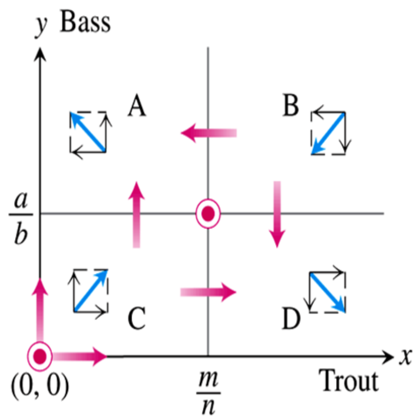
Solving these two equations lead to the points $(x, y) = (0, 0)$ and $(x, y) = (m/n, a/b)$. These points are called *equilibria* or *rest points*.

We note that if $y = a/b$, then $dx/dt = 0$, so the trout population $x(t)$ is constant. Similarly, if $x = m/n$, then $dy/dt = 0$ and the bass population $y(t)$ is constant.



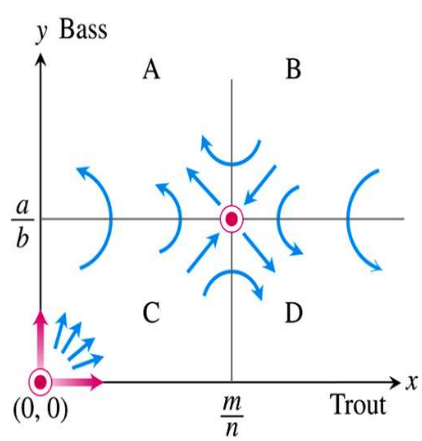
Page 543 Figure 9.26

We now determine the signs of dx/dt and dy/dt throughout the phase plane. When dx/dt is positive, $x(t)$ is increasing and the point is moving to the right in the phase plane. If dx/dt is negative, the point is moving to the left. Likewise, the point is moving upward where dy/dt is positive and downward where dy/dt is negative. Considering the behavior of the point in the four regions of the first quadrant given in Figure 9.26c, we have:



Page 543 Figure 9.29

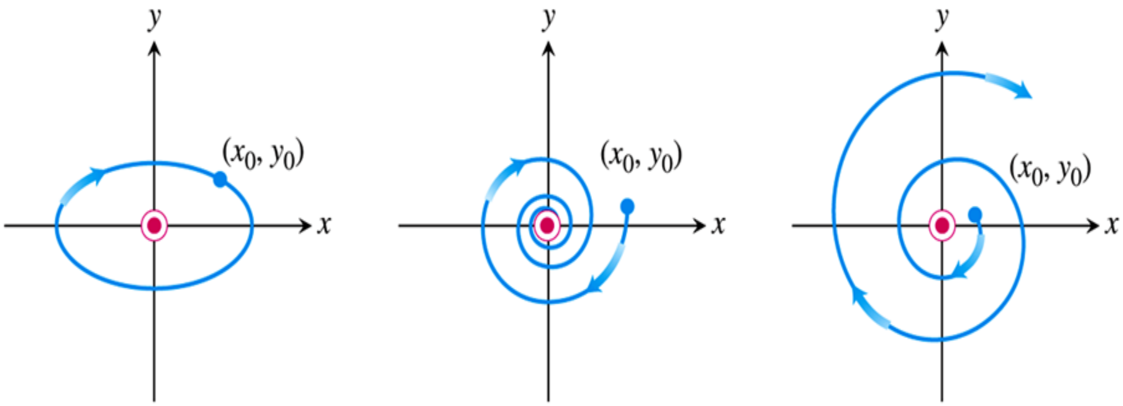
Similar to the results of the previous section, we can trace out trajectories in the phase plane using the above information to get:



Page 543 Figure 9.30

Notice that both of the equilibria are unstable.

Note. Other possible trajectories in the phase plane include:

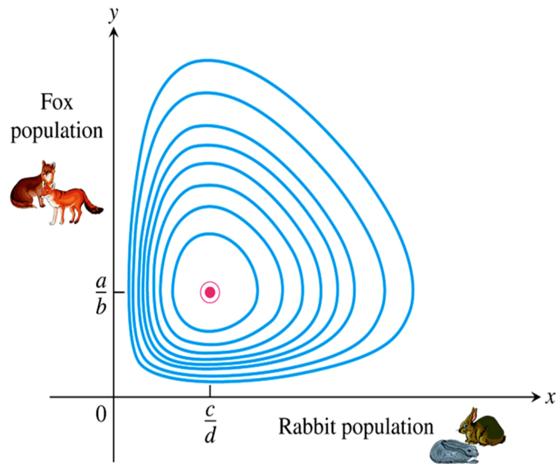


Page 544 Figure 9.33. From left to right these trajectories represent periodic motion, an asymptotically stable equilibrium, and an unstable equilibrium.

Example. Page 546, the Lotka-Volterra Predator-Prey Model. Let $x(t)$ represent the number of rabbits living in a region at time t, and $y(t)$ the number of foxes (which prey on the rabbits) in the same region. Then the model describing the relationships between $x(t)$ and $y(t)$ are:

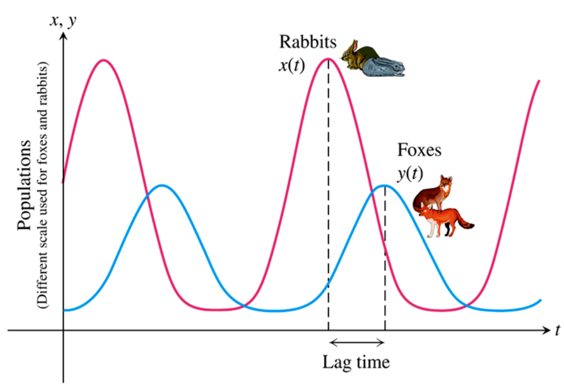
$$\begin{aligned}\frac{dx}{dt} &= (a - by)x,\\ \frac{dy}{dt} &= (-c + dx)y.\end{aligned}$$

An analysis of these equations yields trajectories in the phase plane:



Page 546 Figure 9.37

Graphing $x(t)$ and $y(t)$ as functions of time gives:



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