Calculus 2 Test 2 — Summer 2011

NAME KEY STUDENT NUMBER

SHOW ALL WORK!!! Do not rely on the calculators. Include all necessary symbols (such as equal signs, limits, and constants). The more details you show, the easier it will be to give you partial credit (if needed). Put your final answer in the box provided, or put a box around your final answer. Each numbered problem is worth 12 points.

p459 #20

1. Evaluate $\int t^2 e^{4t}\, dt$.

let $u = t^2$ & $dv = e^{4t}dt$

$du = 2t\,dt$ & $v = \frac{1}{4}e^{4t}$

$= (t^2)\left(\frac{1}{4}e^{4t}\right) - \int\left(\frac{1}{4}e^{4t}\right)(2t\,dt)$

$= \frac{1}{4}t^2e^{4t} - \frac{1}{2}\int te^{4t}dt$

let $u = t$ & $dv = e^{4t}dt$

$du = dt$ & $v = \frac{1}{4}e^{4t}$

$= \frac{1}{4}t^2e^{4t} - \frac{1}{2}\left((t)\left(\frac{1}{4}e^{4t}\right) - \int\frac{1}{4}e^{4t}dt\right)$

$= \boxed{\frac{1}{4}t^2e^{4t} - \frac{1}{8}te^{4t} + \frac{1}{32}e^{4t} + C}$

p466 #11

2. Evaluate $\int \sin^3 x \cos^3 x\, dx$.

$= \int \sin^3 x \cos^2 x \cos x\,dx$

$= \int \sin^3(x)(1 - \sin^2 x)\boxed{\cos x\,dx}$

let $u = \sin(x)$

$du = \cos(x)\,dx$

$= \int u^3(1 - u^2)\,du$

$= \int(u^3 - u^5)\,du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C =$

$\boxed{\frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C}$

3. Evaluate $\int \frac{dx}{\sqrt{9+x^2}}$.

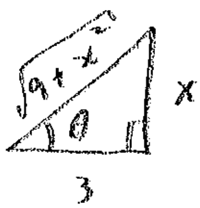
p 470 #1

let $x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$$= \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9+(3\tan\theta)^2}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9+9\tan^2\theta}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9\sec^2\theta}}$$

$$= \int \frac{3\sec^2\theta\, d\theta}{3\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$



$\tan\theta = \frac{x}{3}$

$\sec\theta = \frac{\sqrt{9+x^2}}{3}$

$$\boxed{\ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C}$$

4. (a) What is the *form* of the partial fraction decomposition of the integrand $\frac{x^4+81}{x(x^2+9)^2}$. Notice that you are not asked to find the partial fraction decomposition, but merely to **set up** the form of the partial fractions.

p 479 #26

$$\frac{x^4+81}{x(x^2+9)^2} = \frac{A_1}{x} + \frac{B_1 x + C_1}{(x^2+9)} + \frac{B_2 x + C_2}{(x^2+9)^2}$$

p 479 #21

(b) Evaluate $\int_0^1 \frac{dx}{(x+1)(x^2+1)}$. HINT: The partial fraction decomposition is $\frac{1}{(x+1)(x^2+1)} = \frac{1/2}{x+1} + \frac{-1/2x+1/2}{x^2+1}$.

$$= \int_0^1 \left(\frac{1/2}{x+1} + \frac{-1/2\, x}{x^2+1} + \frac{1/2}{x^2+1}\right) dx = \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx - \frac{1}{2}\int_0^1 \frac{x}{x^2+1}\,dx + \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{u}\,\frac{du}{2} + \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \left(\frac{1}{2}\ln(x+1) - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}(x)\right)\Big|_0^1$$

$$= \left(\frac{1}{2}\ln(2) - \frac{1}{4}\ln(2) + \frac{1}{2}\left(\frac{\pi}{4}\right)\right) - 0$$

$$\boxed{\frac{1}{4}\ln(2) + \frac{\pi}{8}}$$

p 505 #14

5. Evaluate $\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$. Don't write anything wrong!

$$= \int_{-\infty}^{0} \frac{x\,dx}{(x^2+4)^{3/2}} + \int_{0}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}} = \lim_{a\to-\infty} \int_a^0 \frac{x\,dx}{(x^2+4)^{3/2}} + \lim_{b\to\infty} \int_0^b \frac{x\,dx}{(x^2+4)^{3/2}}$$

$$= \lim_{a\to-\infty} \left(\int_{x=a}^{x=0} u^{-3/2} \frac{du}{2} \right) + \lim_{b\to\infty} \left(\int_{x=0}^{x=b} u^{-3/2} \frac{du}{2} \right)$$

let $u = x^2+4$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$

$$= \lim_{a\to-\infty} \left(\frac{-1}{\sqrt{x^2+4}} \Big|_a^0 \right) + \lim_{b\to\infty} \left(\frac{-1}{\sqrt{x^2+4}} \Big|_0^b \right)$$

$$= -\frac{1}{2} + \lim_{a\to\infty} \left(\frac{1}{\sqrt{x^2+4}} \right) - \lim_{b\to\infty} \left(\frac{1}{\sqrt{x^2+4}} \right) + \frac{1}{2} = -\frac{1}{2} + 0 - 0 + \frac{1}{2}$$

$$\boxed{= 0}$$

6. Evaluate $e^{2x}y' + 2e^{2x}y = 2x$. HINT: The solution of the linear equation $\frac{dy}{dx} + P(x)y = Q(x)$ is $y \in \frac{1}{v(x)} \int v(x)Q(x)\,dx$, where $v(x) \in e^{\int P(x)\,dx}$. In the "formula" for $v(x)$, we can simply set v equal to the exponentiation of ANY antiderivative of P.

p 526 #8

Well $y' + \underbrace{2}_{P} y = \underbrace{2e^{-2x}x}_{Q}$ and $\int P(x)\,dx = \int 2\,dx = 2x + C$, so $v(x) = e^{2x}$.

$$y \in \frac{1}{e^{2x}} \int e^{2x} \cdot 2e^{-2x} x\,dx = \frac{1}{e^{2x}} \int 2x\,dx$$

$$= e^{-2x}(x^2 + C)$$

$$\boxed{y = e^{-2x}(x^2 + C)}$$

7. In mechanics, it is common to assume that the force of resistance of a moving object is proportional to the velocity of the object. With velocity as v, mass as m and time as t, we have from Newton's law of motion, Force = mass × acceleration or $m\frac{dv}{dt} = -kv$ or $\frac{dv}{dt} = -\frac{k}{m}v$ where $k > 0$. Assume that when $t = 0$, $v = v_0$. Solve this initial value problem.

p 528

$$\frac{dv}{dt} = -\frac{k}{m}v \Rightarrow \frac{1}{v}\,dv = -\frac{k}{m}\,dt \Rightarrow \int \frac{1}{v}\,dv = \int -\frac{k}{m}\,dt$$

$$\Rightarrow \ln|v| = -\frac{k}{m}t + C \Rightarrow \ln(v) = -\frac{k}{m}t + C \text{ (assume } v>0)$$

$$\Rightarrow e^{\ln(v)} = e^{-(k/m)t + C} = e^{-(k/m)t}e^{C} = v.$$

when $t = 0$, $v(0) = e^{0} \cdot e^{C} = v_0$

$\Rightarrow e^{C} = v_0$ and $\rightarrow$ $\boxed{v = v_0 e^{-(k/m)t}}$

8. Consider the autonomous differential equation $\frac{dy}{dx} = (y+2)(y-3)$. Find the equilibria, construct a phase line, and sketch several solutions. Which equilibria are stable and which are unstable?

p 540 #1

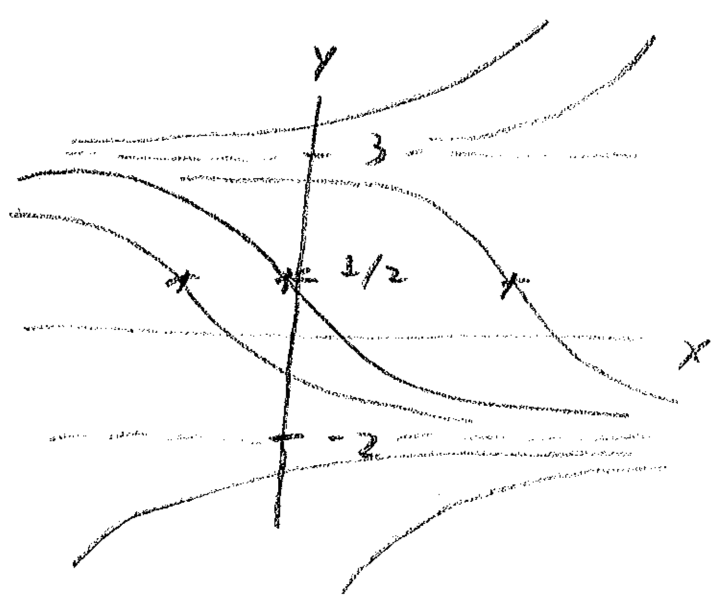
Let $y' = (y+2)(y-3) = 0 \Rightarrow$ $\boxed{y = -2 \text{ and } y = 3 \text{ are eq.}}$

Next, $y'' = \frac{d}{dx}[y^2 - y - 6] = 2y[y'] - y' = (2y-1)y'$

$= (2y-1)(y+2)(y-3)$. So the phase line is:

		-2		$\frac{1}{2}$		3	
y''	$-$		$+$		$-$		$+$
y'	$+$		$-$		$-$		$+$

Some solutions are:



So $y = 3$ is UNSTABLE

$y = -2$ is STABLE.

p 505
#53

Bonus. Consider $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$. Use the Limit Comparison Test to decide whether this integral converges or diverges.

Let $f(x) = \frac{\sqrt{x+1}}{x^2}$ and $g(x) = x^{-3/2}$. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\left(\frac{\sqrt{x+1}}{x^2}\right)}{\left(x^{-3/2}\right)} = \lim_{x\to\infty}\left(\frac{\sqrt{x+1}}{x^2}\right)\left(x^{3/2}\right)$$

$$= \lim_{x\to\infty} \frac{\sqrt{x+1}}{x^{1/2}} = \lim_{x\to\infty}\sqrt{\frac{x+1}{x}} = \lim_{x\to\infty}\sqrt{1+\frac{1}{x}} = \sqrt{1+0} = 1.$$

Next, $\int_1^\infty x^{-3/2}\,dx = \lim_{b\to\infty}\left(\int_1^b x^{-3/2}\,dx\right)$

$\boxed{\text{CONVERGES}}$

$$= \lim_{b\to\infty}\left(-2x^{-1/2}\right)\Big|_1^b = \lim_{b\to\infty}\left(-2b^{-1/2}+2(1)^{-1/2}\right)$$

$$= \lim_{b\to\infty}\left(\frac{-2}{\sqrt{b}}+2\right) = 0+2 = 2.$$ So by the

Limit Comparison Test, $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ converges also.