Calculus 2 Test 3 — Summer 2011

NAME KEY STUDENT NUMBER __________

SHOW ALL WORK!!! Do not rely on the calculators. Include all necessary symbols (such as equal signs and summation signs). Justify every claim you make (such as why a series converges or diverges). The more details you show, the easier it will be to give you partial credit (if needed). Put your final answer in the box provided, or put a box around your final answer. Each numbered problem is worth 12 points.

p 569 #49

1. Consider the series $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$. Does it converge or diverge? If it converges, what is its sum?

This is a geometric series with first term $a = \left(\frac{1}{\sqrt{2}}\right)^0 = 1$ and ratio $r = \frac{1}{\sqrt{2}}$. So the sum is

$$\frac{a}{1-r} = \frac{1}{1-1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}-1}$$

$$\boxed{\frac{\sqrt{2}}{\sqrt{2}-1}}$$

p 575 #7

2. Consider the series $\sum_{n=1}^{\infty}\frac{n}{n^2+4}$. **Use the Integral Test** to determine if the series converges or diverges.

Define $f(x) = \frac{x}{x^2+4}$. Consider $\int_1^{\infty}\left(\frac{x}{x^2+4}\right)dx =$

$$\lim_{b\to\infty}\left(\int_1^b\left(\frac{x}{x^2+4}\right)dx\right) = \lim_{b\to\infty}\left(\frac{1}{2}\ln(x^2+4)\Big|_1^b\right) = \lim_{b\to\infty}\left(\frac{1}{2}\ln(b^2+4) - \frac{1}{2}\ln(5)\right)$$

$= \infty$. So $\int_1^{\infty} f(x)\,dx$ diverges and by the Integral Test, the series diverges.

$\boxed{\text{Diverges}}$

3. Consider the series $\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$. **Use a Comparison Test** to determine if the series converges or diverges.

p580 #9

Let $\sum b_n = \sum \frac{1}{n^2}$ (which converges since it is a p-series with $p = 2 > 1$). Now

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{\left(\frac{n-2}{n^3-n^2+3}\right)}{(1/n^2)} = \lim_{n\to\infty} \left(\frac{n^3-2n^2}{n^3-n^2+3}\right) = 1.$$

So by the Limit Comparison Test, the given series also converges.

$\boxed{\text{Converges}}$

4. Consider the series $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$. **Use the Ratio Test or Root Test** to determine if the series converges or diverges.

p585 #37

By the Ratio Test,

$$\rho = \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \frac{(n+1)!}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{(2n+1)!}{(2n+3)!}$$

$$= \lim_{n\to\infty} \frac{n+1}{(2n+3)(2n+2)} = 0 < 1,$$ so by the Ratio Test,

the series converges.

$\boxed{\text{Converges}}$

p 591 #18

5. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$. Does it converge absolutely, converge conditionally, or diverge?

By the Alternating Series Test, $\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{1}{1+\sqrt{n}}\right) = 0$, so the series converges. For absolute convergence, consider $\sum_{n=1}^{\infty}\left|\frac{(-1)}{1+\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{1+\sqrt{n}}$. Define $\sum b_n = \sum \frac{1}{\sqrt{n}}$ (which diverges since it is a p-series with $p = \frac{1}{2} \le 1$). Now

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\frac{\left(\frac{1}{1+\sqrt{n}}\right)}{(1/\sqrt{n})} = \lim_{n\to\infty}\left(\frac{\sqrt{n}}{1+\sqrt{n}}\right) = 1.$$

So by the Limit Comparison Test, the series $\sum \frac{1}{1+\sqrt{n}}$ also diverges.

So the original series is conditionally convergent. $\boxed{\text{Conditionally Convergent}}$

6. Consider the power series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$. Find the radius of convergence.

p 595 Eg. 3b

We use the Ratio Test to test for absolute convergence.

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left|\frac{(-1)^{(n+1)-1}\,x^{2(n+1)-1}}{2(n+1)-1}\right|}{\left|(-1)^{n-1}\frac{x^{2n-1}}{2n-1}\right|} = \lim_{n\to\infty}\left(\frac{2n-1}{2n+1}\,\frac{|x|^{2n+1}}{|x|^{2n-1}}\right)$$

$= \lim_{n\to\infty}\left(\frac{2n-1}{2n+1}\right)|x|^2 = |x|^2$. So the series converges absolutely if $|x|^2 < 1$ or $|x| < 1$ or $x \in (-1, 1)$. So $R = 1$.

$$\boxed{R = 1}$$

7. Find the Maclaurin series (that is, the Taylor series at $x = 0$) for $f(x) = e^{-x}$. Show details and the computation of the n^{th} derivative.

p.606
#11

Well, $f(x) = e^{-x}$ and $f(0) = e^{-(0)} = 1$

$f'(x) = -e^{-x}$ and $f'(0) = -e^{-(0)} = -1$

$f''(x) = e^{-x}$ and $f''(0) = e^{-(0)} = 1$

$\vdots$

$f^{(n)}(x) = e^{-x}$ and $f^{(n)}(0) = e^{-(0)} = 1$ for n odd

$f^{(n)}(x) = -e^{-x}$ and $f^{(n)}(0) = -e^{-(0)} = -1$ for n even.

So $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} x^n$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} x^n}$$

8. Use series to evaluate $\lim_{x\to 0} \frac{e^x - (1+x)}{x^2}$. HINT: The the series for $f(x) = e^x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

p621
#29

$$\lim_{x\to 0} \frac{e^x - (1+x)}{x^2} = \lim_{x\to 0} \frac{\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right) - (1+x)}{x^2}$$

$$= \lim_{x\to 0} \frac{1 + x + \sum_{n=2}^{\infty}\left(\frac{x^n}{n!}\right) - 1 - x}{x^2} = \lim_{x\to 0} \frac{\sum_{n=2}^{\infty}\left(\frac{x^n}{n!}\right)}{x^2}$$

$$= \lim_{x\to 0} \left(\sum_{n=2}^{\infty}\left(\frac{x^{n-2}}{n!}\right)\right) = \lim_{x\to 0}\left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right) = \frac{1}{2} + 0 = \frac{1}{2}$$

$$\boxed{\frac{1}{2}}$$

Bonus. Consider the series $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$. Does this series converge or diverge?

p 569 #28

Notice $\lim_{n\to\infty} (a_n) = \lim_{n\to\infty} \frac{n(n+1)}{(n+2)(n+3)} = \lim_{n\to\infty} \left(\frac{n^2+n}{n^2+5n+6}\right)$

$\overset{\infty/\infty}{=} \lim_{n\to\infty} \left(\frac{2n+1}{2n+5}\right) \overset{\infty/\infty}{=} \lim_{n\to\infty} \left(\frac{2}{2}\right) = 1 \neq 0.$

So by the Test for Divergence, the series diverges.

$\boxed{\text{Diverge.}}$