Chapter 6. Transcendental Functions and Differential Equations

6.7 Hyperbolic Functions

Note. Recall that cos \(x\) and sin \(x\) are sometimes called the *circular functions*. This is because we can plot a point \(P\) on the circle \(x^2 + y^2 = 1\) by letting \(P = (\cos u, \sin u)\) where \(u\) is twice the area of the sector determined by \(A = (1, 0)\), \(O = (0, 0)\) and \(P\):

Page 529 Figure 6.30b
Note. We can similarly define the *hyperbolic functions*. Consider the hyperbola $x^2 - y^2 = 1$. Choose a point $P$ on the hyperbola and define $u$ as twice the (signed) area determined by the sector $A = (1, 0), O = (0, 0)$, and $P$. Now use the coordinates of $P$ to define the hyperbolic trigonometric functions: $P = (\cosh u, \sinh u)$.
Note. We will use the exponential function to define the hyperbolic trig functions.

Definition. We define

Hyperbolic cosine of $x$: $\cosh x = \frac{e^x + e^{-x}}{2}$

Hyperbolic sine of $x$: $\sinh x = \frac{e^x - e^{-x}}{2}$

Hyperbolic tangent $x$: $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent of $x$: $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant of $x$: $\text{sech } x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

Hyperbolic cosecant of $x$: $\text{csch } x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$
The graphs are:

(a) The hyperbolic sine and its component exponentials.

(b) The hyperbolic cosine and its component exponentials.

(c) The graphs of $y = \tanh x$ and $y = \coth x = 1/\tanh x$.

(d) The graphs of $y = \cosh x$ and $y = \sech x = 1/\cosh x$.

(e) The graphs of $y = \sinh x$ and $y = \csch x = 1/\sinh x$.

Page 521 Figure 6.26
Note. We have the following identities:

\[
\cosh^2 x - \sinh^2 x = 1
\]
\[
\tanh^2 x = 1 - \sech^2 x
\]
\[
\coth^2 x = 1 + \csch^2 x
\]
\[
\sinh 2x = 2 \sinh x \cosh x
\]
\[
\cosh^2 x = \frac{\cosh 2x + 1}{2}
\]
\[
\sinh^2 x = \frac{\cosh 2x - 1}{2}
\]

Example. Page 526 number 2.

Theorem. (Table 6.13) We have the following differentiation properties:

\[
\frac{d}{dx} [\sinh u] = \cosh u \frac{du}{dx}
\]
\[
\frac{d}{dx} [\cosh u] = \sinh u \frac{du}{dx}
\]
\[
\frac{d}{dx} [\tanh u] = \sech^2 u \frac{du}{dx}
\]
\[
\frac{d}{dx} [\coth u] = - \csch^2 u \frac{du}{dx}
\]
\[
\frac{d}{dx} [\sech u] = - \sech u \tanh u \frac{du}{dx}
\]
\[
\frac{d}{dx} [\csch u] = - \csch u \coth u \frac{du}{dx}
\]
Example. Prove some of the results in the above theorem.

Theorem. (Table 6.14) We have the following integral properties:

\[
\int \sinh u \, du = \cosh u + C \\
\int \cosh u \, du = \sinh u + C \\
\int \sech^2 u \, du = \tanh u + C \\
\int \csch^2 u \, du = - \coth u + C \\
\int \sech u \tanh u \, du = - \sech u + C \\
\int \csch u \coth u \, du = - \csch u + C
\]

Proof. These are just the integral versions of the results in Table 6.13.
Q.E.D.

Examples. Page 526 numbers 16 and 22.
Note. Since \( \frac{d}{dx} [\sinh x] = \cosh x > 0 \), then \( \sinh x \) is an INCreasing function and so is one-to-one. The function \( \cosh x \) is not one-to-one as we can see from the graph. The function \( \text{sech } x = 1/ \cosh x \) is also not one-to-one. Therefore, to define the inverse functions of \( \cosh x \) and \( \text{sech } x \), we must restrict the domains.

Definition. We define some inverse hyperbolic trig functions.

Define \( y = \sinh^{-1} x \) if \( x = \sinh y \). (The domain is then \( x \in (-\infty, \infty) \).)

Define \( y = \cosh^{-1} x \) if \( x = \cosh y \) and \( y \in [0, \infty) \). (The domain is then \( x \in [1, \infty) \).)

Define \( y = \text{sech}^{-1} x \) if \( x = \text{sech } y \) and \( y \in [0, \infty) \). (The domain is then \( x \in (0, 1] \).)
Note. The graphs of the above defined three inverse hyperbolic trig functions are:

Note/Definition. The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

\[ y = \tanh^{-1} x, \quad y = \coth^{-1} x, \quad y = \csch^{-1} x. \]

The graphs of these functions are:
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Page 524 Figure 6.28

**Theorem.** We can express the inverse hyperbolic trig functions in terms of the natural logarithm function as follows:

\[
\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \ x \in (-\infty, \infty).
\]

\[
\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \ x \in [1, \infty).
\]

\[
\tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right), \ x \in (-1, 1).
\]

\[
\text{sech}^{-1} x = \ln \left( \frac{1 + \sqrt{1 - x^2}}{x} \right), \ x \in (0, 1].
\]

\[
\text{csch}^{-1} x = \ln \left( \frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|} \right), \ x \in (-\infty, 0) \cup (0, \infty).
\]
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\[ \coth^{-1} x = \frac{1}{2} \ln \frac{x + 1}{x - 1}, \; x \in (-\infty, -1) \cup (1, \infty). \]

Note. (Table 6.15) We can verify the following identities:

\[
\begin{align*}
\sech^{-1} x &= \cosh^{-1} \frac{1}{x} \\
\csch^{-1} x &= \sinh^{-1} \frac{1}{x} \\
\coth^{-1} x &= \tanh^{-1} \frac{1}{x}
\end{align*}
\]

Theorem. (Table 6.16) The inverse hyperbolic trig functions are differentiated as follows:

\[
\begin{align*}
\frac{d}{dx} [\sinh^{-1}] &= \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx} \\
\frac{d}{dx} [\cosh^{-1}] &= \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \; u \in (1, \infty) \\
\frac{d}{dx} [\tanh^{-1}] &= \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \; u \in (-1, 1) \\
\frac{d}{dx} [\coth^{-1}] &= \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \; u \in (-\infty, -1) \cup (1, \infty) \\
\frac{d}{dx} [\sech^{-1}] &= \frac{-1}{u\sqrt{1 - u^2}} \frac{du}{dx}, \; u \in (0, 1) \\
\frac{d}{dx} [\csch^{-1}] &= \frac{-1}{|u|\sqrt{1 + u^2}} \frac{du}{dx}, \; u \in (-\infty, 0) \cup (0, \infty)
\end{align*}
\]
Example. Page 527 number 34.

Theorem. (Table 6.17) We have the following integrals involving inverse hyperbolic trig functions:

\[ \int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1} \left( \frac{u}{a} \right) + C, \ a > 0 \]

\[ \int \frac{du}{\sqrt{a^2 + u^2}} = \cosh^{-1} \left( \frac{u}{a} \right) + C, \ u > a > 0 \]

\[ \int \frac{du}{u^2 - u^2} = \begin{cases} \frac{1}{a} \tanh^{-1} \left( \frac{u}{a} \right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a} \coth^{-1} \left( \frac{u}{a} \right) + C & \text{if } u^2 > a^2 \end{cases} \]

\[ \int \frac{du}{u \sqrt{a^2 - u^2}} = -\frac{1}{a} \operatorname{sech}^{-1} \left( \frac{u}{a} \right) + C, \ 0 < u < a \]

\[ \int \frac{du}{u \sqrt{a^2 + u^2}} = -\frac{a}{a} \csch^{-1} \left| \frac{u}{a} \right| + C, \ u \neq 0 \]

Examples. Page 527 number 74, page 528 number 79.

Corrected 1/18/2020