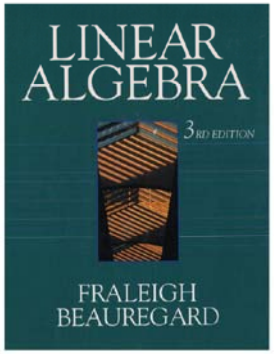
Linear Algebra

Chapter 1. Vectors, Matrices, and Linear Systems

Section 1.1. Vectors in Euclidean Spaces—Proofs of Theorems



Page 16 Number 10

Page 16 Number 10. Compute the linear combination $3\vec{u} + \vec{v} - \vec{w}$ where $\vec{u} = [1, 2, 1, 0]$, $\vec{v} = [-2, 0, 1, 6]$, and $\vec{w} = [3, -5, 1, -2]$.

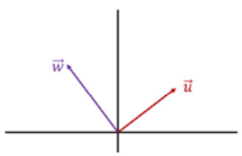
Solution. We have

$$\begin{aligned} 3\vec{u} + \vec{v} - \vec{w} &= 3[1,2,1,0] + [-2,0,1,6] - [3,-5,1,-2] \\ &= [3(1), 3(2), 3(1), 3(0)] + [-2,0,1,6] - [3,-5,1,-2] \\ &\quad \text{by Definition 1.1(3), “Scalar Multiplication”} \\ &= [3,6,3,0] + [-2,0,1,6] - [3,-5,1,-2] \text{ simplifying} \\ &= [3+(-2), 6+0, 3+1, 0+6] - [3,-5,1,-2] \\ &\quad \text{by Definition 1.1(1), “Vector Addition”} \\ &= [1,6,4,6] - [3,-5,1,-2] \text{ simplifying} \\ &= [1-(3), 6-(-5), 4-(1), 6-(-2)] \\ &\quad \text{by Definition 1.1(2), “Vector Subtraction”} \\ &= [-2, 11, 3, 8] \text{ simplifying.} \end{aligned}$$

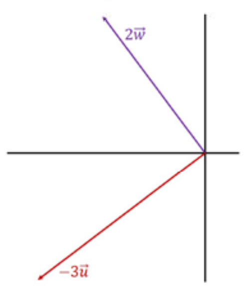
So we conclude $\boxed{3\vec{u} + \vec{v} - \vec{w} = [-2, 11, 3, 8].}$ □

Page 16 Number 14

Page 16 Number 14. Reproduce the vectors in this figure and draw an arrow representing $-3\vec{u} + 2\vec{w}$.

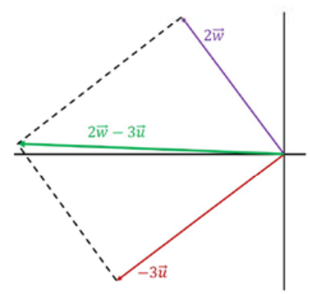


Solution. From Definition 1.1(3), “Scalar Multiplication,” and the geometric interpretation of vectors (see the class notes, pages 2, 3, and 4) we represent $-3\vec{u}$ and $2\vec{w}$ as:



Page 16 Number 14

Then by the parallelogram property of addition:



□

Page 17 Number 40(a)

Page 17 Number 40(a). Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and let r, s be scalars in $\mathbb{R}$. Prove (A1): $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$.

Proof. Since $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, by Definition 1.A, "Vectors in $\mathbb{R}^n$," we have that $\vec{u} = [u_1, u_2, \ldots, u_n]$, $\vec{v} = [v_1, v_2, \ldots, v_n]$, and $\vec{w} = [w_1, w_2, \ldots, w_n]$ where all u_i, v_i, and w_i are real numbers. Then

$$\begin{aligned}(\vec{u}+\vec{v})+\vec{w} &= ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n] \\ &= [u_1+v_1, u_2+v_2, \ldots, u_n+v_n] + [w_1, w_2, \ldots, w_n] \\ &\quad \text{by Definition 1.1(1), "Vector Addition"} \\ &= [(u_1+v_1)+w_1, (u_2+v_2)+w_2, \ldots, (u_n+v_n)+w_n] \\ &\quad \text{by Definition 1.1(1), "Vector Addition"} \\ &= [u_1+(v_1+w_1), u_2+(v_2+w_2), \ldots, u_n+(v_n+w_n)] \\ &\quad \text{since addition of real numbers is associative}\end{aligned}$$

Page 17 Number 40(a) (continued)

Page 17 Number 40(a). Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and let r, s be scalars in $\mathbb{R}$. Prove (A1): $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$.

Proof (continued). ...

$$\begin{aligned}(\vec{u}+\vec{v})+\vec{w} &= [u_1+(v_1+w_1), u_2+(v_2+w_2), \ldots, u_n+(v_n+w_n)] \\ &= [u_1, u_2, \ldots, u_n] + [v_1+w_1, v_2+w_2, \ldots, v_n+w_n] \\ &\quad \text{by Definition 1.1(1), "Vector Addition"} \\ &= [u_1, u_2, \ldots, u_n] + ([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n]) \\ &\quad \text{by Definition 1.1(1), "Vector Addition"} \\ &= \vec{u}+(\vec{v}+\vec{w}).\end{aligned}$$

□

Page 17 Number 41(a)

Page 17 Number 41(a). Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let r be a scalar in $\mathbb{R}$. Prove (S1): $r(\vec{v}+\vec{w}) = r\vec{v}+r\vec{w}$.

Proof. Since $\vec{v}, \vec{w} \in \mathbb{R}^n$, by Definition 1.A, "Vectors in $\mathbb{R}^n$," we have that $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ where all v_i and w_i are real numbers. Then

$$\begin{aligned}r(\vec{v}+\vec{w}) &= r([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n]) \\ &= r[v_1+w_1, v_2+w_2, \ldots, v_n+w_n] \\ &\quad \text{by Definition 1.1(1), "Vector Addition"} \\ &= [r(v_1+w_1), r(v_2+w_2), \ldots, r(v_n+w_n)] \\ &\quad \text{by Definition 1.1(3), "Scalar Multiplication"} \\ &= [rv_1+rw_1, rv_2+rw_2, \ldots, rv_n+rw_n] \\ &\quad \text{since multiplication distributes} \\ &\quad \text{over addition in the real numbers...}\end{aligned}$$

Page 17 Number 41(a) (continued)

Page 17 Number 41(a). Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let r be a scalar in $\mathbb{R}$. Prove (S1): $r(\vec{v}+\vec{w}) = r\vec{v}+r\vec{w}$.

Proof (continued). ...

$$\begin{aligned}r(\vec{v}+\vec{w}) &= [rv_1+rw_1, rv_2+rw_2, \ldots, rv_n+rw_n] \\ &= [rv_1, rv_2, \ldots, rv_n] + [rw_1, rw_2, \ldots, rw_n] \\ &\quad \text{by Definition 1.1(1), "Vector Addition"} \\ &= r[v_1, v_2, \ldots, v_n] + r[w_1, w_2, \ldots, w_n] \\ &\quad \text{by Definition 1.1(3), "Scalar Multiplication"} \\ &= r\vec{v}+r\vec{w}.\end{aligned}$$

□

Page 16 Number 22

Page 16 Number 22. Find all scalars c (if any) such that the vector $[c^2, -4]$ is parallel to the vector $[1, -2]$.

Solution. By Definition 1.2, two nonzero vectors are parallel if one is a scalar multiple of the other, say $[c^2, -4] = r[1, -2]$ for scalar $r \in \mathbb{R}$. Then by Definition 1.1(3), "Scalar Multiplication," $[c^2, -4] = [r, -2r]$. So we need both $c^2 = r$ and $-4 = -2r$. Since $-4 = -2r$ then we must have $r = 2$. With $r = 2$ and $c^2 = r = 2$ we must have that either $\boxed{c = \sqrt{2} \text{ or } c = -\sqrt{2}.}$ □

Page 16 Number 28

Page 16 Number 28. Find all scalars c (if any) such that the vector $\vec{\imath} + c\vec{\jmath} + (c-1)\vec{k}$ is in the span of $\vec{\imath} + 2\vec{\jmath} + \vec{k}$ and $3\vec{\imath} + 6\vec{\jmath} + 3\vec{k}$.
Solution. By Definition 1.4, the span of $\vec{\imath} + 2\vec{\jmath} + \vec{k}$ and $3\vec{\imath} + 6\vec{\jmath} + 3\vec{k}$ is the set of all linear combinations of these two vectors. So the question becomes: For which $c \in \mathbb{R}$ is
$\vec{\imath} + c\vec{\jmath} + (c-1)\vec{k} = r_1(\vec{\imath} + 2\vec{\jmath} + \vec{k}) + r_2(3\vec{\imath} + 6\vec{\jmath} + 3\vec{k})$ for some $r_1, r_2 \in \mathbb{R}$?
If this holds, $\vec{\imath} + c\vec{\jmath} + (c-1)\vec{k} = (r_1 + 3r_2)\vec{\imath} + (2r_1 + 6r_2)\vec{\jmath} + (r_1 + 3r_2)\vec{k}$.
So we need $c \in \mathbb{R}$ such that

$$\begin{aligned} 1 &= r_1 + 3r_2 && (1) \\ c &= 2r_1 + 6r_2 && (2) \\ c - 1 &= r_1 + 3r_2 && (3) \end{aligned}$$

Multiplying (1) by 2 gives $2 = 2r_1 + 6r_2$. Combining this with (2) we see that we need $c = 2$. With $c = 2$, equation (3) gives $1 = r_1 + 3r_2$ which is (1). Therefore all three equations (1), (2), and (3) are satisfied when $\boxed{c = 2.}$ We can take $r_1 = 1$ and $r_2 = 0$, for example. □