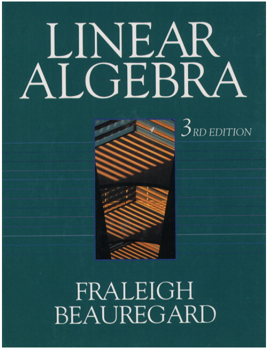
Linear Algebra

Chapter 1. Vectors, Matrices, and Linear Systems

Section 1.1. Vectors in Euclidean Spaces—Proofs of Theorems



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Page 16 Number 10. Compute the linear combination $3\vec{u}+\vec{v}-\vec{w}$ where $\vec{u}=[1,2,1,0]$, $\vec{v}=[-2,0,1,6]$, and $\vec{w}=[3,-5,1,-2]$.

Solution. We have

$$\begin{aligned}
3\vec{u}+\vec{v}-\vec{w} &= 3[1,2,1,0]+[-2,0,1,6]-[3,-5,1,-2] \\
&= [3(1),3(2),3(1),3(0)]+[-2,0,1,6]-[3,-5,1,-2] \\
&\quad \text{by Definition 1.1(3), ``Scalar Multiplication''} \\
&= [3,6,3,0]+[-2,0,1,6]-[3,-5,1,-2] \text{ simplifying}
\end{aligned}$$

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Page 16 Number 10. Compute the linear combination $3\vec{u} + \vec{v} - \vec{w}$ where $\vec{u} = [1, 2, 1, 0]$, $\vec{v} = [-2, 0, 1, 6]$, and $\vec{w} = [3, -5, 1, -2]$.

Solution. We have

$$\begin{aligned}
3\vec{u} + \vec{v} - \vec{w} &= 3[1, 2, 1, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2] \\
&= [3(1), 3(2), 3(1), 3(0)] + [-2, 0, 1, 6] - [3, -5, 1, -2] \\
&\quad \text{by Definition 1.1(3), ``Scalar Multiplication''} \\
&= [3, 6, 3, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2] \text{ simplifying} \\
&= [3 + (-2), 6 + 0, 3 + 1, 0 + 6] - [3, -5, 1, -2] \\
&\quad \text{by Definition 1.1(1), ``Vector Addition''} \\
&= [1, 6, 4, 6] - [3, -5, 1, -2] \text{ simplifying}
\end{aligned}$$

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&\quad \text{by Definition 1.1(3), ``Scalar Multiplication''} \\
&= [3,6,3,0]+[-2,0,1,6]-[3,-5,1,-2] \text{ simplifying} \\
&= [3+(-2),6+0,3+1,0+6]-[3,-5,1,-2] \\
&\quad \text{by Definition 1.1(1), ``Vector Addition''} \\
&= [1,6,4,6]-[3,-5,1,-2] \text{ simplifying} \\
&= [1-(3),6-(-5),4-(1),6-(-2)] \\
&\quad \text{by Definition 1.1(2), ``Vector Subtraction''} \\
&= [-2,11,3,8] \text{ simplifying.}
\end{aligned}$$

So we conclude $\boxed{3\vec{u}+\vec{v}-\vec{w}=[-2,11,3,8].}$ □

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Solution. We have

$$\begin{aligned}
3\vec{u} + \vec{v} - \vec{w} &= 3[1, 2, 1, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2] \\
&= [3(1), 3(2), 3(1), 3(0)] + [-2, 0, 1, 6] - [3, -5, 1, -2] \\
&\quad \text{ by Definition 1.1(3), “Scalar Multiplication”} \\
&= [3, 6, 3, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2] \text{ simplifying} \\
&= [3 + (-2), 6 + 0, 3 + 1, 0 + 6] - [3, -5, 1, -2] \\
&\quad \text{ by Definition 1.1(1), “Vector Addition”} \\
&= [1, 6, 4, 6] - [3, -5, 1, -2] \text{ simplifying} \\
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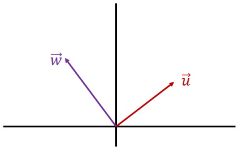
\end{aligned}$$

So we conclude $\boxed{3\vec{u} + \vec{v} - \vec{w} = [-2, 11, 3, 8].}$ □

Page 16 Number 14

Page 16 Number 14. Reproduce the vectors in this figure and draw an arrow representing $-3\vec{u} + 2\vec{w}$.



Solution. From Definition 1.1(3), "Scalar Multiplication," and the geometric interpretation of vectors (see the class notes, pages 2, 3, and 4) we represent $-3\vec{u}$ and $2\vec{w}$ as:

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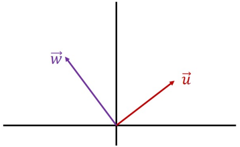

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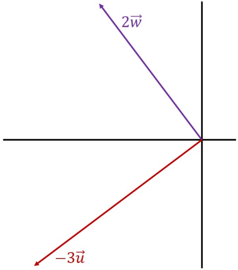

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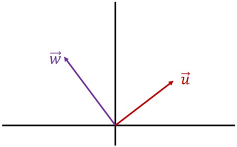

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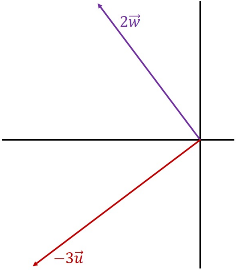

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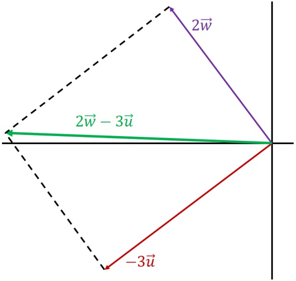
Then by the parallelogram property of addition:



□

Page 17 Number 40(a)

Page 17 Number 40(a). Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and let r, s be scalars in $\mathbb{R}$. Prove (A1): $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

Proof. Since $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, by Definition 1.A, "Vectors in $\mathbb{R}^n$," we have that $\vec{u} = [u_1, u_2, \ldots, u_n]$, $\vec{v} = [v_1, v_2, \ldots, v_n]$, and $\vec{w} = [w_1, w_2, \ldots, w_n]$ where all u_i, v_i, and w_i are real numbers.

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$$\begin{aligned} (\vec{u} + \vec{v}) + \vec{w} &= ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n] \\ &= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] + [w_1, w_2, \ldots, w_n] \end{aligned}$$

by Definition 1.1(1), "Vector Addition"

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(\vec{u} + \vec{v}) + \vec{w} &= ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n] \\
&= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] + [w_1, w_2, \ldots, w_n] \\
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&= [(u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n] \\
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$$
\begin{aligned}
(\vec{u} + \vec{v}) + \vec{w} &= ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n] \\
&= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] + [w_1, w_2, \ldots, w_n] \\
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&= [(u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n] \\
&\quad \text{by Definition 1.1(1), "Vector Addition"} \\
&= [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)] \\
&\quad \text{since addition of real numbers is associative}
\end{aligned}
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Proof (continued). ...

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(\vec{u} + \vec{v}) + \vec{w} &= [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)] \\
&= [u_1, u_2, \ldots, u_n] + [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n] \\
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□

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Page 17 Number 41(a). Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let r be a scalar in $\mathbb{R}$. Prove (S1): $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$.

Proof. Since $\vec{v}, \vec{w} \in \mathbb{R}^n$, by Definition 1.A, "Vectors in $\mathbb{R}^n$," we have that $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ where all v_i and w_i are real numbers.

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$$\begin{aligned} r(\vec{v} + \vec{w}) &= r([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n]) \\ &= r[v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n] \\ &\quad \text{by Definition 1.1(1), "Vector Addition"} \end{aligned}$$

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&\quad \text{by Definition 1.1(1), "Vector Addition"} \\
&= [r(v_1 + w_1), r(v_2 + w_2), \ldots, r(v_n + w_n)] \\
&\quad \text{by Definition 1.1(3), "Scalar Multiplication"} \\
&= [rv_1 + rw_1, rv_2 + rw_2, \ldots, rv_n + rw_n] \\
&\quad \text{since multiplication distributes} \\
&\quad \text{over addition in the real numbers...}
\end{aligned}$$

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&\quad \text{by Definition 1.1(1), "Vector Addition"} \\
&= [r(v_1 + w_1), r(v_2 + w_2), \ldots, r(v_n + w_n)] \\
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&= [rv_1 + rw_1, rv_2 + rw_2, \ldots, rv_n + rw_n] \\
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\end{aligned}$$

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Proof (continued). ...

$$\begin{aligned} r(\vec{v} + \vec{w}) &= [rv_1 + rw_1, rv_2 + rw_2, \ldots, rv_n + rw_n] \\ &= [rv_1, rv_2, \ldots, rv_n] + [rw_1, rw_2, \ldots, rw_n] \\ &\quad \text{by Definition 1.1(1), “Vector Addition”} \\ &= r[v_1, v_2, \ldots, v_n] + r[w_1, w_2, \ldots, w_n] \\ &\quad \text{by Definition 1.1(3), “Scalar Multiplication”} \\ &= r\vec{v} + r\vec{w}. \end{aligned}$$

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□

Page 16 Number 22

Page 16 Number 22. Find all scalars c (if any) such that the vector $[c^2, -4]$ is parallel to the vector $[1, -2]$.

Solution. By Definition 1.2, two nonzero vectors are parallel if one is a scalar multiple of the other, say $[c^2, -4] = r[1, -2]$ for scalar $r \in \mathbb{R}$. Then by Definition 1.1(3), "Scalar Multiplication," $[c^2, -4] = [r, -2r]$.

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Solution. By Definition 1.2, two nonzero vectors are parallel if one is a scalar multiple of the other, say $[c^2, -4] = r[1, -2]$ for scalar $r \in \mathbb{R}$. Then by Definition 1.1(3), "Scalar Multiplication," $[c^2, -4] = [r, -2r]$. So we need both $c^2 = r$ and $-4 = -2r$. Since $-4 = -2r$ then we must have $r = 2$. With $r = 2$ and $c^2 = r = 2$ we must have that either $\boxed{c = \sqrt{2} \text{ or } c = -\sqrt{2}.}$ □

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Solution. By Definition 1.4, the span of $\vec{\imath} + 2\vec{\jmath} + \vec{k}$ and $3\vec{\imath} + 6\vec{\jmath} + 3\vec{k}$ is the set of all linear combinations of these two vectors. So the question becomes: For which $c \in \mathbb{R}$ is
$\vec{\imath} + c\vec{\jmath} + (c-1)\vec{k} = r_1(\vec{\imath} + 2\vec{\jmath} + \vec{k}) + r_2(3\vec{\imath} + 6\vec{\jmath} + 3\vec{k})$ for some $r_1, r_2 \in \mathbb{R}$?

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If this holds, $\vec{\imath} + c\vec{\jmath} + (c-1)\vec{k} = (r_1 + 3r_2)\vec{\imath} + (2r_1 + 6r_2)\vec{\jmath} + (r_1 + 3r_2)\vec{k}$.
So we need $c \in \mathbb{R}$ such that

$$\begin{aligned} 1 &= r_1 + 3r_2 && (1) \\ c &= 2r_1 + 6r_2 && (2) \\ c - 1 &= r_1 + 3r_2 && (3) \end{aligned}$$

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