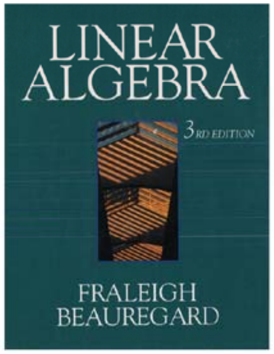
Linear Algebra

Chapter 1. Vectors, Matrices, and Linear Systems
Section 1.2. The Norm and Dot Product—Proofs of Theorems



Page 31 Number 8

Page 31 Number 8. Find the unit vector parallel to $\vec{w} = [-2, -1, 3]$ which has the opposite direction.

Solution. If we divide $\vec{w}$ by the scalar $\|\vec{w}\| > 0$, we get a vector of length 1 (i.e., a unit vector; this process is called *normalizing* a vector). Such a vector is in the same direction as $\vec{w}$ (by Definition 1.2 of "parallel and same direction"). By Definition 1.5, "Vector Norm," we have $\|\vec{w}\| = \sqrt{(-2)^2 + (-1)^2 + (3)^2} = \sqrt{4+1+9} = \sqrt{14}$, so $\dfrac{\vec{w}}{\|\vec{w}\|} = \dfrac{1}{\sqrt{14}}[-2,-1,3] = \left[\dfrac{-2}{\sqrt{14}}, \dfrac{-1}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right]$ is a unit vector in the same direction as $\vec{w}$. To get a unit vector in the opposite direction, by Definition 1.2, we simply multiply by -1 and take $-\vec{w}/\|\vec{w}\|$ as the desired vector: $-\dfrac{\vec{w}}{\|\vec{w}\|} = -\left[\dfrac{-2}{\sqrt{14}}, \dfrac{-1}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right] = \boxed{\left[\dfrac{2}{\sqrt{14}}, \dfrac{1}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}\right].}$ □

Page 31 Number 12

Page 31 Number 12. Find the angle between $\vec{u} = [-1, 3, 4]$ and $\vec{v} = [2, 1, -1]$.

Solution. We have by definition that the desired angle is $\cos^{-1}\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$. Now by Definition 1.5, "Vector Norm," $\|\vec{u}\| = \sqrt{(-1)^2 + (3)^2 + (4)^2} = \sqrt{1+9+16} = \sqrt{26}$ and $\|\vec{v}\| = \sqrt{(2)^2 + (1)^2 + (-1)^2} = \sqrt{4+1+1} = \sqrt{6}$. Also, by Definition 1.6, "Dot Product," $\vec{u}\cdot\vec{v} = [-1,3,4]\cdot[2,1,-1] = (-1)(2)+(3)(1)+(4)(-1) = -2+3-4 = -3$. So the angle between $\vec{u}$ and $\vec{v}$ is $\cos^{-1}\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \cos^{-1}\dfrac{-3}{\sqrt{26}\sqrt{6}} = \boxed{\cos^{-1}\dfrac{-3}{\sqrt{156}}.}$ We can use a calculator to approximate the true answer to find that the angle is roughly 103.90°.

Page 33 Number 42(b)

Page 33 Number 42(b). Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$. Prove the Distributive Law: $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$.

Proof. Since $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, then by our first definition in Section 1.1, we have that $\vec{u} = [u_1, u_2, \dots, u_n]$, $\vec{v} = [v_1, v_2, \dots, v_n]$, and $\vec{w} = [w_1, w_2, \dots, w_n]$ where all u_i, v_i, w_i are real numbers. Then

$$\begin{aligned}
\vec{u}\cdot(\vec{v}+\vec{w}) &= [u_1, u_2, \dots, u_n]\cdot([v_1, v_2, \dots, v_n] + [w_1, w_2, \dots, w_n]) \\
&= [u_1, u_2, \dots, u_n]\cdot[v_1+w_1, v_2+w_2, \dots, v_n+w_n] \\
&\quad \text{by Definition 1.1.(1), "Vector Addition"} \\
&= u_1(v_1+w_1) + u_2(v_2+w_2) + \cdots + u_n(v_n+w_n) \\
&\quad \text{by Definition 1.6, "Dot Product"} \\
&= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \cdots + u_nv_n + u_nw_n \\
&\quad \text{since multiplication distributes over addition in } \mathbb{R}
\end{aligned}$$

...

Page 33 Number 42(b) (continued)

Page 33 Number 42(b). Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$. Prove the Distributive Law: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

Proof (continued). ...

$$\begin{aligned}
\vec{u} \cdot (\vec{v} + \vec{w}) &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \cdots + u_nv_n + u_nw_n \\
&= (u_1v_1 + u_2v_2 + \cdots + u_nv_n) + (u_1w_1 + u_2w_2 + \cdots + u_nw_n) \\
&\quad \text{since addition is commutative and associative in } \mathbb{R} \\
&= [u_1, u_2, \ldots, u_n] \cdot [v_1, v_2, \ldots, v_n] \\
&\quad +[u_1, u_2, \ldots, u_n] \cdot [w_1, w_2, \ldots, w_n] \\
&\quad \text{by Definition 1.6, “Dot Product”} \\
&= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}.
\end{aligned}$$

□

Page 31 Number 14

Page 31 Number 14. Find the value of x such that $[x, -3, 5]$ is perpendicular to $\vec{u} = [-1, 3, 4]$.

Solution. By the definition of perpendicular (see page 4 of the class notes) we want x such that $[x, -3, 5] \cdot [-1, 3, 4] = 0$. Now

$$[x, -3, 5] \cdot [-1, 3, 4] = (x)(-1) + (-3)(3) + (5)(4) = -x - 9 + 20 = -x + 11.$$

So to get a dot product of 0 we must have $\boxed{x = 11.}$ □

Page 31 Number 16

Page 31 Number 16. Find a nonzero vector in $\mathbb{R}^3$ which is perpendicular to both $\vec{u} = [-1, 3, 4]$ and $\vec{v} = [2, 1, -1]$.

Solution. Let the desired vector be $\vec{w} = [w_1, w_2, w_3]$. By the definition of perpendicular (see page 4 of the class notes) we need $\vec{w} \cdot \vec{u} = 0$ and $\vec{w} \cdot \vec{v} = 0$. This gives

$$\vec{w} \cdot \vec{u} = [w_1, w_2, w_3] \cdot [-1, 3, 4]$$

$$= (w_1)(-1) + (w_2)(3) + (w_3)(4) = -w_1 + 3w_2 + 4w_3 = 0$$

and

$$\vec{w} \cdot \vec{v} = [w_1, w_2, w_3] \cdot [2, 1, -1]$$

$$= (w_1)(2) + (w_2)(1) + (w_3)(-1) = 2w_1 + w_2 - w_3 = 0.$$

So we need $w_1, w_2, w_3 \in \mathbb{R}$ that satisfy both:

$$\begin{aligned} -w_1 + 3w_2 + 4w_3 &= 0 \quad (1) \\ 2w_1 + w_2 - w_3 &= 0. \quad (2) \end{aligned}$$

...

Page 31 Number 16 (continued)

Solution (continued). ...

$$\begin{aligned} -w_1 + 3w_2 + 4w_3 &= 0 \quad (1) \\ 2w_1 + w_2 - w_3 &= 0. \quad (2) \end{aligned}$$

Adding 2 times equation (1) to equation (2) gives $0w_1 + 7w_2 + 7w_3 = 0$. So we *can* take $w_2 = 1$ and $w_3 = -1$. Plugging these values into equation (1) gives $-w_1 + 3(1) + 4(-1) = 0$ and so $-w_1 - 1 = 0$ or $w_1 = -1$. So *a* choice for w_1, w_2, w_3 is $w_1 = -1$, $w_2 = 1$, and $w_3 = -1$. That is, we can choose $\boxed{\vec{w} = [w_1, w_2, w_3] = [-1, 1, -1].}$ (In fact, any nonzero multiple of this choice of $\vec{w}$ is also correct.)

Let's check the orthogonality:

$$\vec{w} \cdot \vec{u} = [-1, 1, -1] \cdot [-1, 3, 4] = (-1)(-1) + (1)(3) + (-1)(4) = 1 + 3 - 4 = 0$$

and

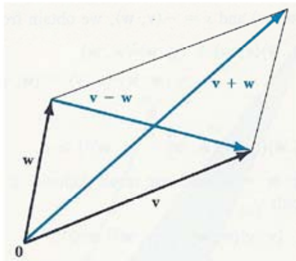
$$\vec{w} \cdot \vec{v} = [-1, 1, -1] \cdot [2, 1, -1] = (-1)(2) + (1)(1) + (-1)(-1) = -2 + 1 + 1 = 0.$$

So, by the definition of perpendicular, $\vec{w}$ is perpendicular to both $\vec{u}$ and $\vec{v}$, as required. □

Page 26 Example 7

Page 26 Example 7. Prove that the sum of the squares of the lengths of the diagonals of a parallelogram in $\mathbb{R}^n$ is equal to the sum of the squares of the lengths of the sides. This is the *parallelogram relation* or the *parallelogram law*.

Proof. Let two of the sides of the parallelogram be determined by vectors $\vec{v}$ and $\vec{w}$ in standard position:



Then the lengths of the sides of the parallelogram are $\|\vec{v}\|$, $\|\vec{v}\|$, $\|\vec{w}\|$, and $\|\vec{w}\|$; the lengths of the diagonals are $\|\vec{v}+\vec{w}\|$ and $\|\vec{v}-\vec{w}\|$.

Page 26 Example 7 (continued)

Proof (continued). Expressing the squares of norms using dot products as in Note 1.2.A:

$$\begin{aligned}
\|\vec{v}+\vec{w}\|^2+\|\vec{v}-\vec{w}\|^2 &= (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})+(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) \\
&= (\vec{v}\cdot\vec{v}+2\vec{v}\cdot\vec{w}+\vec{w}\cdot\vec{w}) \\
&\quad +(\vec{v}\cdot\vec{v}-2\vec{v}\cdot\vec{w}+\vec{w}\cdot\vec{w}) \\
&\quad \text{ by Theorem 1.3(D1) and (D2),} \\
&\quad \text{"Commutivity and Distribution of Dot Product"} \\
&= 2\vec{v}\cdot\vec{v}+2\vec{w}\cdot\vec{w}=2\|\vec{v}\|^2+2\|\vec{w}\|^2.
\end{aligned}$$

So the sum of the squares of the lengths of the diagonals, $\|\vec{v}+\vec{w}\|^2+\|\vec{v}-\vec{w}\|^2$, equals the sum of the squares of the lengths of the sides, $\|\vec{v}\|^2+\|\vec{v}\|^2+\|\vec{w}\|^2+\|\vec{w}\|^2=2\|\vec{v}\|^2+2\|\vec{w}\|^2$. □

Theorem 1.4

Theorem 1.4. Schwarz's Inequality.
Let $\vec{v},\vec{w}\in\mathbb{R}^n$. Then

$$|\vec{v}\cdot\vec{w}|\le\|\vec{v}\|\|\vec{w}\|.$$

Proof. Let $\vec{v},\vec{w}\in\mathbb{R}^n$ and let r and s be any scalars in $\mathbb{R}$. Then $\|r\vec{v}+s\vec{w}\|\ge 0$ by Theorem 1.2(1), "Positivity of the Norm," and so

$$\begin{aligned}
0 &\le \|r\vec{v}+s\vec{w}\|^2=(r\vec{v}+s\vec{w})\cdot(r\vec{v}+s\vec{w}) \text{ by Note 1.2.A} \\
&= (r\vec{v})\cdot(r\vec{v})+2(r\vec{v})\cdot(s\vec{w})+(s\vec{w})\cdot(s\vec{w}) \\
&\quad \text{by Theorem 1.3(D1) and (D2), "Commutivity and} \\
&\quad \text{Distribution of Dot Products"} \\
&= r^2\vec{v}\cdot\vec{v}+2rs\vec{v}\cdot\vec{w}+s^2\vec{w}\cdot\vec{w} \\
&\quad \text{by Theorem 1.3(D3), "Homogeneity of Dot Products"} \\
&= r^2\|\vec{v}\|^2+2rs\vec{v}\cdot\vec{w}+s^2\|\vec{w}\|^2 \text{ by Note 1.2.A.}
\end{aligned}$$

Theorem 1.4 (continued)

Theorem 1.4. Schwarz's Inequality.
Let $\vec{v},\vec{w}\in\mathbb{R}^n$. Then $|\vec{v}\cdot\vec{w}|\le\|\vec{v}\|\|\vec{w}\|$.

Proof (continued). Since this holds for all scalars $r,s\in\mathbb{R}$, we can let $r=\|\vec{w}\|^2$ and $s=-\vec{v}\cdot\vec{w}$ and hence

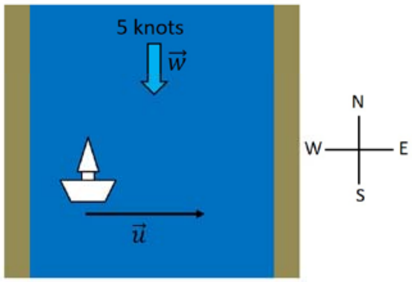
$$\begin{aligned}
0 &\le r^2\|\vec{v}\|^2+2rs\vec{v}\cdot\vec{w}+s^2\|\vec{w}\|^2 \\
&= \|\vec{w}\|^4\|\vec{v}\|^2-2\|\vec{w}\|^2(\vec{v}\cdot\vec{w})^2+(\vec{v}\cdot\vec{w})^2\|\vec{w}\|^2 \\
&= \|\vec{w}\|^4\|\vec{v}\|^2-\|\vec{w}\|^2(\vec{v}\cdot\vec{w})^2 \\
&= \|\vec{w}\|^2(\|\vec{w}\|^2\|\vec{v}\|^2-(\vec{v}\cdot\vec{w})^2). \quad (*)
\end{aligned}$$

If $\|\vec{w}\|=0$ then $\vec{w}=\vec{0}$ by Theorem 1.3(D4), "Positivity of the Dot Product," and then $\vec{v}\cdot\vec{w}=\vec{v}\cdot\vec{0}=0$ so that $0=|\vec{v}\cdot\vec{w}|\le\|\vec{v}\|\|\vec{w}\|=\|\vec{v}\|0=0$ and Schwarz's Inequality holds. If $\|\vec{w}\|\ne 0$ then from $(*)$, dividing both sides by $\|\vec{w}\|^2$, we have that $\|\vec{v}\|^2\|\vec{w}\|^2-(\vec{v}\cdot\vec{w})^2\ge 0$. That is, $(\vec{v}\cdot\vec{w})^2\le\|\vec{v}\|^2\|\vec{w}\|^2$ and so $\sqrt{(\vec{v}\cdot\vec{w})^2}\le\sqrt{\|\vec{v}\|^2\|\vec{w}\|^2}$ or $|\vec{v}\cdot\vec{w}|\le\|\vec{v}\|\|\vec{w}\|$, as claimed. □

Page 31 Number 36

Page 31 Number 36. The captain of a barge wishes to get to a point directly across a straight river that runs north to south. If the current flows directly downstream at 5 knots and the barge steams at 13 knots, in what direction should the captain steer the barge?

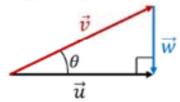
Solution. Consider the diagram:



We need the barge to have a velocity $\vec{v}$ such that $\vec{v} + \vec{w}$ results in a vector $\vec{u}$ that runs east-west.

Page 31 Number 36 (continued)

Solution (continued). By the parallelogram property of the addition of vectors (see Figure 1.1.5, page 5) we have:



where $\vec{w} = [0, -5]$ knots and $\vec{u} = [u_1, u_2] = [u_1, 0]$ knots. So with $\vec{v} = [v_1, v_2]$, we have $\vec{v} + \vec{w} = \vec{u}$ or $[v_1, v_2] + [0, -5] = [u_1, 0]$ or $[v_1, v_2 - 5] = [u_1, 0]$. Hence $v_2 = 5$ knots. Since $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} = \sqrt{v_1^2 + (5)^2} = 13$ knots then $\sqrt{v_1^2 + 25} = 13$ and $v_1^2 + 25 = 169$ or $v_1^2 = 144$ (knots^2) or $v_1 = 12$ knots. Then $u_1 = v_1 = 12$ knots and so $\vec{u} = [12, 0]$ knots. Notice from the right triangle determined by $\vec{u}$, $\vec{w}$, and $\vec{v}$ we have $\cos\theta = \|\vec{u}\|/\|\vec{v}\| = 12/13$ and so $\theta = \cos^{-1}(12/13)$. So

$$\boxed{\text{the captain should steer the barge } \theta = \cos^{-1}(12/13) \text{ upstream.}}$$ □