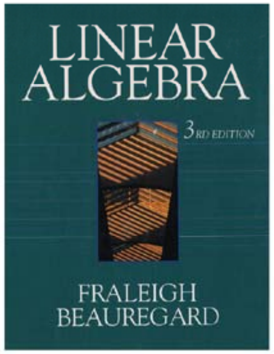
# Linear Algebra

**Chapter 1. Vectors, Matrices, and Linear Systems**

Section 1.3. Matrices and Their Algebra—Proofs of Theorems








## Page 46 Number 16

**Page 46 Number 16.** Let $B = \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 \\ 0 & 6 \\ -3 & 2 \end{bmatrix}$.

Compute $BC$ and $CB$.

**Solution.** First, notice that $B$ is $2 \times 3$ and $C$ is $3 \times 2$, so both products actually exist, $BC$ is $2 \times 2$, and $CB$ is $3 \times 3$. By Definition 1.8, "Matrix Product," we have

$$\begin{aligned} BC &= \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 6 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} (4)(2)+(1)(0)+(-2)(-3) & (4)(-1)+(1)(6)+(-2)(2) \\ (5)(2)+(-1)(0)+(3)(-3) & (5)(-1)+(-1)(6)+(3)(2) \end{bmatrix} \\ &= \begin{bmatrix} 8+0+6 & -4+6-4 \\ 10+0-9 & -5-6+6 \end{bmatrix} = \boxed{\begin{bmatrix} 14 & -2 \\ 1 & -5 \end{bmatrix}}. \end{aligned}$$





## Page 46 Number 16 (continued)

**Page 46 Number 16.** Let $B = \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 \\ 0 & 6 \\ -3 & 2 \end{bmatrix}$.

Compute $BC$ and $CB$.

**Solution (continued).** By Definition 1.8, "Matrix Product," we have

$$\begin{aligned} CB &= \begin{bmatrix} 2 & -1 \\ 0 & 6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} (2)(4)+(-1)(5) & (2)(1)+(-1)(-1) & (2)(-2)+(-1)(3) \\ (0)(4)+(6)(5) & (0)(1)+(6)(-1) & (0)(-2)+(6)(3) \\ (-3)(4)+(2)(5) & (-3)(1)+(2)(-1) & (-3)(-2)+(2)(3) \end{bmatrix} \\ &= \begin{bmatrix} 8-5 & 2+1 & -4-3 \\ 0+30 & 0-6 & 0+18 \\ -12+10 & -3-2 & 6+6 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 3 & -7 \\ 30 & -6 & 18 \\ -2 & -5 & 12 \end{bmatrix}}. \quad \square \end{aligned}$$





## Page 46 Number 6

**Page 46 Number 6.** Let $A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}$ and

$B = \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix}$. Compute $4A - 2B$.

**Solution.** Notice that $A$ and $B$ are both $2 \times 3$ matrices so the sum actually exists. By Definition 1.9/1.10, "Matrix Sum and Scalar Multiplication,"

$$\begin{aligned} 4A - 2B &= 4\begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} - 2\begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4(-2) & 4(1) & 4(3) \\ 4(4) & 4(0) & 4(-1) \end{bmatrix} + \begin{bmatrix} -2(4) & -2(1) & -2(-2) \\ -2(5) & -2(-1) & -2(3) \end{bmatrix} \\ &= \begin{bmatrix} -8 & 4 & 12 \\ 16 & 0 & -4 \end{bmatrix} + \begin{bmatrix} -8 & -2 & 4 \\ -10 & 2 & -6 \end{bmatrix} \\ &= \begin{bmatrix} -8+(-8) & 4+(-2) & 12+4 \\ 16+(-10) & 0+2 & -4+(-6) \end{bmatrix} = \boxed{\begin{bmatrix} -16 & 2 & 16 \\ 6 & 2 & -10 \end{bmatrix}} \end{aligned}$$

$\square$

## Page 47 Number 33

**Page 47 Number 33.** Let $A$, $B$, and $C$ be matrices where the products $(AB)C$ and $A(BC)$ are defined. Then matrix multiplication is associative: $A(BC) = (AB)C$.

**Proof.** Let $A = [a_{ij}]$ be $m \times n$, $B = [b_{ij}]$ be $n \times s$, and $C = [c_{ij}]$ be $s \times t$. The $(i,j)$ entry of $BC$ is $\sum_{k=1}^{s} b_{ik}c_{kj}$ and so the $(k,j)$ entry of $BC$ is $\sum_{\ell=1}^{s} b_{k\ell}c_{\ell j}$. Therefore the $(i,j)$ entry of $A(BC)$ is

$$\sum_{k=1}^{n} a_{ik}\left(\sum_{\ell=1}^{s} b_{k\ell}c_{\ell j}\right) = \sum_{\ell=1}^{s}\left(\sum_{k=1}^{n} a_{ik}b_{k\ell}\right)c_{\ell j} = \sum_{k=1}^{s}\left(\sum_{\ell=1}^{n} a_{i\ell}b_{\ell k}\right)c_{kj}$$

where the second equality holds by interchanging dummy variables $\ell$ and $k$. Now $\sum_{\ell=1}^{n} a_{i\ell}b_{\ell k}$ is the $(i,k)$ entry of $AB$, and so the last sum is the $(i,j)$ entry of $(AB)C$. Therefore $A(BC) = (AB)C$. □





## Example 1.3.A

**Example 1.3.A.** Show that $\mathcal{I}A = A\mathcal{I} = A$ for $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $\mathcal{I}$ is $3 \times 3$.

**Solution.** By Definition 1.8, "Matrix Product," we have

$$\mathcal{I}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(1)+(0)(4)+(0)(7) & (1)(2)+(0)(5)+(0)(8) & (1)(3)+(0)(6)+(0)(9) \\ (0)(1)+(1)(4)+(0)(7) & (0)(2)+(1)(5)+(0)(8) & (0)(3)+(1)(6)+(0)(9) \\ (0)(1)+(0)(4)+(1)(7) & (0)(2)+(0)(5)+(1)(8) & (0)(3)+(0)(6)+(1)(9) \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 2+0+0 & 3+0+0 \\ 0+4+0 & 0+5+0 & 0+6+0 \\ 0+0+7 & 0+0+8 & 0+0+9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A.$$





## Example 1.3.A (continued)

**Example 1.3.A.** Show that $\mathcal{I}A = A\mathcal{I} = A$ for $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $\mathcal{I}$ is $3 \times 3$.

**Solution (continued).** By Definition 1.8, "Matrix Product," we have

$$A\mathcal{I} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(1)+(2)(0)+(3)(0) & (1)(0)+(2)(1)+(3)(0) & (1)(0)+(2)(0)+(3)(1) \\ (4)(1)+(5)(0)+(6)(0) & (4)(0)+(5)(1)+(6)(0) & (4)(0)+(5)(0)+(6)(1) \\ (7)(1)+(8)(0)+(9)(0) & (7)(0)+(8)(1)+(9)(0) & (7)(0)+(8)(0)+(9)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+2+0 & 0+0+3 \\ 4+0+0 & 0+5+0 & 0+0+6 \\ 7+0+0 & 0+8+0 & 0+0+9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A.$$

□