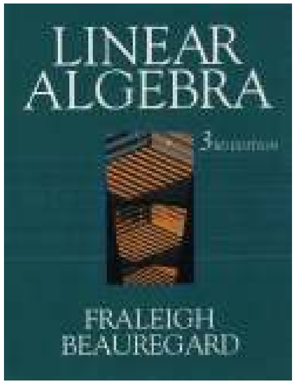
Linear Algebra

Chapter 1. Vectors, Matrices, and Linear Systems

Section 1.4. Solving Systems of Linear Equations—Proofs of Theorems



Example 1.4.A

Example 1.4.A. Solve the system

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & x_3 & = & 4 \\ & & x_2 & - & x_3 & = & -1 \\ & & & & x_3 & = & 3. \end{array} \qquad \begin{array}{r}(1)\\(2)\\(3)\end{array}$$

Solution. From (3) we have $x_3 = 3$. So from (2) we have $x_2 - (3) = -1$ or $x_2 = 2$. Then from (1) we have $x_1 + 3(2) - (3) = 4$ or $x_1 = 1$. So the solution is $\boxed{x_1 = 1, x_2 = 2, x_3 = 3.}$ □

Example 1.4.B

Example 1.4.B. Is this system consistent or inconsistent:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & - & x_2 & + & 3x_3 & = & 1 \\ 3x_1 & & & + & 2x_3 & = & 3? \end{array}$$

Solution. We create the augmented matrix for the system: $\left[\begin{array}{rrr|r} 2 & 1 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 3 & 0 & 2 & 3 \end{array}\right]$. We now use elementary row operations to put the augmented matrix in row-echelon form. We have

$$\left[\begin{array}{rrr|r} 2 & 1 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 3 & 0 & 2 & 3 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 1 \\ 3 & 0 & 2 & 3 \end{array}\right]$$

Example 1.4.B (continued 1)

Solution (continued).

$$\underset{R_3 \to R_3 - 3R_1}{\underset{\sim}{R_2 \to R_2 - 2R_1}} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 2-2(1) & 1-2(-1) & -1-2(3) & 1-2(1) \\ 3-3(1) & 0-3(-1) & 2-3(3) & 3-3(1) \end{array}\right]$$

$$= \left[\begin{array}{rrr|r} 1 & -1 & 3 & 1 \\ 0 & 3 & -7 & -1 \\ 0 & 3 & -7 & 0 \end{array}\right] \qquad (*)$$

$$\underset{\sim}{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 3 & -7 & -1 \\ 0-(0) & 3-(3) & -7-(-7) & 0-(-1) \end{array}\right]$$

$$= \left[\begin{array}{rrr|r} 1 & -1 & 3 & 1 \\ 0 & 3 & -7 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]. \qquad (**)$$

Example 1.4.B (continued 2)

Example 1.4.B. Is this system consistent or inconsistent:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & - & x_2 & + & 3x_3 & = & 1 \\ 3x_1 & & & + & 2x_3 & = & 3? \end{array}$$

Solution (continued). Now by Theorem 1.6 (Invariance of Solution Sets) we see that the original system has the same solution (if one exists) as each of the systems associated with with any of these augmented matrices. We see that we have a problem in $(*)$ since the second and third rows imply that $3x_2 - 7x_3 = -1$ and $3x_2 - 7x_3 = 0$. Of course, both of these cannot be true so this tells us that there is no solution. Alternatively, the augmented matrix in $(**)$ is in row-echelon form (by Definition 1.12) and the third row of $(**)$ implies that $0 = 1$, which of course is not the case and so the original system has no solution and is $\boxed{\text{inconsistent.}}$ □

Example 1.4.C

Example 1.4.C. Is this system consistent or inconsistent:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & - & x_2 & + & 3x_3 & = & 1 \\ 3x_1 & & & + & 2x_3 & = & 2? \end{array}$$

(HINT: This system has multiple solutions. Express the solutions in terms of an unknown parameter r).

Solution. We take the same approach as in the previous example. The augmented matrix is similar to the one in the previous example and we perform the same row operations (so we give less arithmetic details). So

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 3 & 0 & 2 & 2 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 1 \\ 3 & 0 & 2 & 2 \end{array}\right]$$

$$\underset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 3 & -7 & -1 \\ 0 & 3 & -7 & -1 \end{array}\right]$$

Example 1.4.C (continued 1)

Solution (continued).

$$\underset{R_3 \to R_3 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 3 & -7 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Again, the augmented matrix is in row-echelon form. It corresponds to the equations

$$\begin{array}{rcrcrcll} x_1 & - & x_2 & + & 3x_3 & = & 1 & (1) \\ & & 3x_2 & - & 7x_3 & = & -1 & (2) \\ & & & & 0 & = & 0. & (3) \end{array}$$

From (2) we have $3x_2 = -1 + 7x_3$ or $x_2 = -\frac{1}{3} + \frac{7}{3}x_3$. We can then substitute this into (1) to get $x_1 - \left(-\frac{1}{3} + \frac{7}{3}x_3\right) + 3x_3 = 1$ or $x_1 = \frac{2}{3} - \frac{2}{3}x_3$. This gives the system of equations:

$$\begin{array}{rcrcl} x_1 & = & 2/3 & - & (2/3)x_3 \\ x_2 & = & -1/3 & + & (7/3)x_3 \\ x_3 & = & & & x_3. \end{array}$$

Example 1.4.C (continued 2)

Solution (continued). We introduce a parameter r for x_3 where r can be any real number. We then have

$$\begin{array}{rcrcl} x_1 & = & 2/3 & - & (2/3)r \\ x_2 & = & -1/3 & + & (7/3)r \\ x_3 & = & & & r. \end{array}$$

We can check to see that for any $r \in \mathbb{R}$, this is a solution to each of the original equations:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & x_3 & = & 2(2/3 - (2/3)r) + (-1/3 + (7/3)r) - (r) = 1 \\ x_1 & - & x_2 & + & 3x_3 & = & (2/3 - (2/3)r) - (-1/3 + (7/3)r) + 3(r) = 1 \\ 3x_1 & & & + & 2x_3 & = & 3(2/3 - (2/3)r) + 2(r) = 2. \end{array}$$

So the original system is $\boxed{\text{consistent}}$ and for any $r \in \mathbb{R}$ a solution is given by $x_1 = 2/3 - (2/3)r$, $x_2 = -1/3 + (7/3)r$, $x_3 = r$. Expressed as a vector equation, we have $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2/3 \\ 7/3 \\ 1 \end{bmatrix}$. □

Page 68 Number 2(a)

Page 68 Number 2(a). Use elementary row operations to put $\begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & 3 \\ -1 & -3 & 0 \end{bmatrix}$ in row-echelon form (REF).

Solution. To make the arithmetic easier, we move the -1 in position $(3,1)$ to position $(1,1)$ using the row operation $R_1 \leftrightarrow R_3$; we then get zeros below the pivot at position $(1,1)$ using Step (2b) of the previous note. Then we'll deal with the second column. So

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & 3 \\ -1 & -3 & 0 \end{bmatrix} \underset{\sim}{\overset{R_1 \leftrightarrow R_3}{}} \begin{bmatrix} -1 & -3 & 0 \\ 4 & 8 & 3 \\ 2 & 4 & -2 \end{bmatrix}$$

$$\underset{R_3 \to R_3 + 2R_1}{\overset{R_2 \to R_2 + 4R_1}{\sim}} \begin{bmatrix} -1 & -3 & 0 \\ 4+4(-1) & 8+4(-3) & 3+4(0) \\ 2+2(-1) & 4+2(-3) & -2+2(0) \end{bmatrix}$$

Page 68 Number 2(a) (continued)

Solution (continued).

$$= \begin{bmatrix} -1 & -3 & 0 \\ 0 & -4 & 3 \\ 0 & -2 & -2 \end{bmatrix} \underset{\sim}{\overset{R_2 \leftrightarrow R_3}{}} \begin{bmatrix} -1 & -3 & 0 \\ 0 & -2 & -2 \\ 0 & -4 & 3 \end{bmatrix} \text{ (for ease of computations)}$$

$$\underset{\sim}{\overset{R_3 \to R_3 - 2R_2}{}} \begin{bmatrix} -1 & -3 & 0 \\ 0 & -2 & -2 \\ 0-2(0) & -4-2(-2) & 3-2(-2) \end{bmatrix} = \boxed{\begin{bmatrix} -1 & -3 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 7 \end{bmatrix}}.$$

This matrix satisfies Definition 1.12 and so is in REF. A word of warning: A row-echelon form of a matrix is not unique! For example, we could perform the elementary row operations $R_1 \to -R_1$, $R_2 \to R_2/(-2)$, and $R_3 \to R_3/7$ (to make all pivots 1) and get $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ as an alternative row echelon form of the given matrix. □

Page 69 Number 16(a)

Page 69 number 16(a). Consider

$$\begin{array}{rcrcrcr} 2x & + & y & - & 3z & = & 0 \\ 6x & + & 3y & - & 8z & = & 0 \\ 2x & - & y & + & 5z & = & -4. \end{array}$$

Put the augmented matrix in row-echelon form and use back substitution to solve.

Solution. We have

$$\left[\begin{array}{rrr|r} 2 & 1 & -3 & 0 \\ 6 & 3 & -8 & 0 \\ 2 & -1 & 5 & -4 \end{array}\right]$$

$$\underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - 3R_1}{\sim}} \left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 6-3(2) & 3-3(1) & -8-3(-3) & 0-3(0) \\ 2-(2) & -1-(1) & 5-(-3) & -4-(0) \end{array}\right]$$

Page 69 Number 16(a) (continued)

Solution (continued).

$$= \left[\begin{array}{rrr|r} 2 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 8 & -4 \end{array}\right] \underset{\sim}{\overset{R_2 \leftrightarrow R_3}{}} \left[\begin{array}{rrr|r} 2 & 1 & -3 & 0 \\ 0 & -2 & 8 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

So this matrix is in row echelon form (by Definition 1.12) and the associated system of equations for this matrix is

$$\begin{array}{rcrcrcrr} 2x & + & y & - & 3z & = & 0 & (1) \\ & - & 2y & + & 8z & = & -4 & (2) \\ & & & & z & = & 0. & (3) \end{array}$$

By back substitution, (3) gives $z = 0$. Then (2) gives $-2y + 8(0) = -4$ or $y = 2$. From (1) we have $2x + (2) - 3(0) = 0$ or $x = -1$. So the (unique) solution to the system of equations is $\boxed{x = -1,\ y = 2,\ z = 0.}$ □

Page 68 Number 2(b)

Page 68 Number 2(b). Use elementary row operations to put $\begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & 3 \\ -1 & -3 & 0 \end{bmatrix}$ in reduced row-echelon form (RREF).

Solution. In Number 2(a) we saw that

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & 3 \\ -1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the pivots are all 1 in this new REF matrix, we only need to apply elementary row operations to get 0's above the pivots. We have

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & 3 \\ -1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Page 68 Number 2(b) (continued)

Solution (continued).

$$\underset{R_1 \to R_1 - 3R_2}{\sim} \begin{bmatrix} 1-3(0) & 3-3(1) & 0-3(1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\substack{R_1 \to R_1 + 3R_3 \\ R_2 \to R_2 - R_3}}{\sim} \begin{bmatrix} 1+3(0) & 0+3(0) & -3+3(1) \\ 0-(0) & 1-(0) & 1-(1) \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

This is in reduced row echelon form by the previous definition. Notice that requiring the pivots to all be 1 in a RREF matrix guarantees that there is a unique RREF matrix which is row equivalent to any given matrix. □

Page 69 Number 16(b)

Page 69 number 16(b). Consider

$$\begin{aligned} 2x + y - 3z &= 0 \\ 6x + 3y - 8z &= 0 \\ 2x - y + 5z &= -4. \end{aligned}$$

Put the augmented matrix in reduced row-echelon form and solve.

Solution. In Number 16(a) we saw that the augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 6 & 3 & -8 & 0 \\ 2 & -1 & 5 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 0 & -2 & 8 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

So we need to continue with elementary row operations to make all pivots 1 and make all entries above pivots 0.

Page 69 Number 16(b) (continued 1)

Solution (continued). We do so in an order that avoids the use of fractions:

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 0 & -2 & 8 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right] \underset{R_2 \to R_2/(-2)}{\sim} \left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{R_1 \to R_1 - R_2}{\sim} \left[\begin{array}{ccc|c} 2-(0) & 1-(1) & -3-(-4) & 0-(2) \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 1 & -2 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{\substack{R_1 \to R_1 - R_3 \\ R_2 \to R_2 + 4R_3}}{\sim} \left[\begin{array}{ccc|c} 2-(0) & 0-(0) & 1-(1) & -2-(0) \\ 0+4(0) & 1+4(0) & -4+4(1) & 2+4(0) \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Page 69 Number 16(b) (continued 2)

Solution (continued).

$$= \left[\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{R_1 \to R_1/2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

So associated with this augmented matrix is a system of equations that allows us to just read off the solution: $\boxed{x = -1,\ y = 2,\ z = 0.}$ By Theorem 1.16 (Invariance of Solution Sets) this is the solution to the original system of equations. Notice that this is the same solution as obtained in Number 16(a), though we have avoided back substitution by performing more elementary row operations. □

Theorem 1.7

Theorem 1.7. Solutions of $A\vec{x} = \vec{b}$.
Let $A\vec{x} = \vec{b}$ be a linear system and let $[A \mid \vec{b}] \sim [H \mid \vec{c}]$ where H is in row-echelon form.
1. The system $A\vec{x} = \vec{b}$ is inconsistent if and only if $[H \mid \vec{c}]$ has a row with all entries equal to 0 to the left of the partition and a nonzero entry to the right of the partition.
2. If $A\vec{x} = \vec{b}$ is consistent and every column of H contains a pivot, the system has a unique solution.
3. If $A\vec{x} = \vec{b}$ is consistent and some column of H has no pivot, the system has infinitely many solutions, with as many free variables as there are pivot-free columns of H.

Proof. First, if $[H \mid \vec{c}]$ has a row, say row i, of all entries of 0 to the left of the partition and a nonzero entry c_i to the right of the partition, then this row corresponds to the equation $0x_1 + 0x_2 + \cdots + 0x_n = c_i$ in the system $H\vec{x} = \vec{c}$. By Theorem 1.6, the solution sets of $A\vec{x} = \vec{b}$ and $H\vec{x} = \vec{c}$ are the same.

Theorem 1.7 (continued 1)

Proof (continued). No values of $x_1, x_2, \ldots, x_n$ make $0x_1 + 0x_2 + \cdots + 0x_n = c_i \neq 0$, so the solution set of $H\vec{x} = \vec{c}$ is empty and hence the solution set of $A\vec{x} = \vec{b}$ is empty. That is, $A\vec{x} = \vec{b}$ has no solution and so is inconsistent.

Now suppose that $[H \mid \vec{c}]$ has no row with all entries 0 to the left of the partition and a nonzero entry to the right. We'll show that in this case $A\vec{x} = \vec{b}$ has a solution and so is consistent, completing the proof of part (1). If a row of $[H \mid \vec{c}]$ is all zeros on both sides of the partition, then this corresponds to the equation $0x_1 + 0x_2 + \cdots + 0x_n = 0$ which is satisfied by all $x_1, x_2, \ldots, x_n$ and so this equation contributes no information in determining a solution to $A\vec{x} = \vec{b}$. So we can create matrix $[H' \mid \vec{c}']$ by eliminating all rows of 0's in matrix $[H \mid \vec{c}]$. So every row of H' contains a pivot.

Theorem 1.7 (continued 2)

Proof (continued). If every column of H' (and hence of H) contains a pivot then each x_j is uniquely determined and so the system is consistent and has a unique solution and (2) follows. If $A\vec{x} = \vec{b}$ is consistent and the jth column of H' (and hence of H) contains no pivot then x_j can take on any value (it is a free variable, as in Example 1.4.C). The other x_i, where column i contains a pivot, can then be determined in terms of these free variables and so (3) holds.

Notice that when (2) holds there is a unique solution and when (3) holds there are multiple solutions. In either case, the system is consistent and now (1) follows. □

Page 71 Number 52

Page 71 Number 52. Proof for Row Interchange (Theorem 1.8).

Suppose E results from interchanging Row i and Row j in $\mathcal{I}$: $\mathcal{I} \overset{R_i \leftrightarrow R_j}{\sim} E$. Then the kth row of E is $[0, 0, \ldots, 0, 1, 0, \ldots, 0]$ where

(1) for $k \notin \{i, j\}$ the nonzero entry is the kth entry,

(2) for $k = i$ the nonzero entry is the jth entry, and

(3) for $k = j$ the nonzero entry is the ith entry.

Let $A = [a_{ij}]$, $E = [e_{ij}]$, and $B = [b_{ij}] = EA$. The kth row of B is $[b_{k1}, b_{k2}, \ldots, b_{kn}]$ and

$$b_{k\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell}.$$

Now if $k \notin \{i, j\}$ then all e_{kp} are 0 except for $p = k$ and

$$b_{k\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell} = e_{kk} a_{k\ell} = (1) a_{k\ell} = a_{k\ell}.$$

Page 71 Number 52 (continued)

Proof (continued). Therefore for $k \notin \{i, j\}$, the kth row of B is the same as the kth row of A. If $k = i$ then all e_{kp} are 0 except for $p = j$ and

$$b_{k\ell} = b_{i\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell} = e_{kj} a_{j\ell} = (1) a_{j\ell} = a_{j\ell}$$

and the ith row of B is the same as the jth row of A. Similarly, if $k = j$ then all e_{kp} are 0 except for $p = i$ and

$$b_{k\ell} = b_{j\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell} = e_{ki} a_{i\ell} = (1) a_{i\ell} = a_{i\ell}$$

and the jth row of B is the same as the ith row of A. Therefore

$$B = EA \text{ and } A \overset{R_i \leftrightarrow R_j}{\sim} B,$$

as claimed. □

Page 71 Number 54

Page 71 Number 54. Proof for Row Addition (Theorem 1.8).

Suppose E results from adding s times Row j to Row i in $\mathcal{I}$: $\mathcal{I} \overset{R_i \to R_i + sR_j}{\sim} E$. Then the kth row of E is the same as the kth row of $\mathcal{I}$ for $k \neq i$, and the ith row of E is $[0, 0, \ldots, 0, 1, 0, \ldots, 0, s, 0, \ldots, 0, 0]$ (or $[0, 0, \ldots, 0, s, 0, \ldots, 0, 1, 0, \ldots, 0, 0]$) where the ith component is 1 and the jth component is s and all other components are 0. Let $A = [a_{ij}]$, $E = [e_{ij}]$, and $B = [b_{ij}] = EA$. The kth row of B is $[b_{k1}, b_{k2}, \ldots, b_{kn}]$ and

$$b_{k\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell}.$$

For $k \neq i$,

$$b_{k\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell} = e_{kk} a_{k\ell} = (1) a_{k\ell} = a_{k\ell}.$$

Therefore for $k \neq i$, the kth row of B is the same as the kth row of A.

Page 71 Number 54 (continued)

Proof (continued). If $k = i$

$$b_{k\ell} = b_{i\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell} = e_{ii} a_{i\ell} + e_{ij} a_{j\ell} = (1) a_{i\ell} + (s) a_{j\ell} = a_{i\ell} + s a_{j\ell}$$

and the ith row of B is the same as the ith row of A. Therefore

$$B = EA \text{ and } A \overset{R_i \to R_i + sR_j}{\sim} B,$$

as claimed. □

Example 1.4.D

Example 1.4.D. Multiply some 3×3 matrix A by $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to swap Row 1 and Row 2.

Solution. We have

$$EA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} (0)(1)+(1)(4)+(0)(7) & (0)(2)+(1)(5)+(0)(8) & (0)(3)+(1)(6)+(1)(9) \\ (1)(1)+(0)(4)+(0)(7) & (1)(2)+(0)(5)+(0)(8) & (1)(3)+(0)(6)+(0)(9) \\ (0)(1)+(0)(4)+(1)(7) & (0)(2)+(0)(5)+(1)(8) & (0)(3)+(0)(6)+(1)(9) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}. \quad \square$$

Page 70 Number 44

Page 70 Number 44. Find a matrix C such that

$$C \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -6 \end{bmatrix}.$$

Solution. We see that C must be 3×3, so let $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$.

Then we need

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -6 \end{bmatrix}$$

or

$$\begin{bmatrix} c_{11}+3c_{12}+4c_{13} & 2c_{11}+4c_{12}+2c_{13} \\ c_{21}+3c_{22}+4c_{23} & 2c_{21}+4c_{22}+2c_{23} \\ c_{31}+3c_{32}+4c_{33} & 2c_{31}+4c_{32}+2c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -6 \end{bmatrix}.$$

Page 70 Number 44 (continued 1)

Solution (continued). We can treat this as three systems of equations based on the rows here:

$$\begin{array}{c} c_{11}+3c_{12}+4c_{13}=1 \\ 2c_{11}+4c_{12}+2c_{13}=2 \end{array} \qquad \begin{array}{c} c_{21}+3c_{22}+4c_{23}=0 \\ 2c_{21}+4c_{22}+2c_{23}=-2 \end{array} \qquad \begin{array}{c} c_{31}+3c_{32}+4c_{33}=0 \\ 2c_{31}+4c_{32}+2c_{33}=-6. \end{array}$$

These three systems of equations have associated augmented matrices:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 2 & 4 & 2 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 4 & 2 & -2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 4 & 2 & -6 \end{array}\right].$$

We put each in RREF:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 2 & 4 & 2 & 2 \end{array}\right] \overset{R_2 \to R_2/2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 1 & 2 & 1 & 1 \end{array}\right] \overset{R_2 \to R_2 - R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & -1 & -3 & 0 \end{array}\right]$$

$$\overset{R_2 \to -R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & 1 & 3 & 0 \end{array}\right] \overset{R_1 \to R_1 - 3R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 3 & 0 \end{array}\right] \quad (1)$$

Page 70 Number 44 (continued 2)

Solution (continued).

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 4 & 2 & -2 \end{array}\right] \overset{R_2 \to R_2/2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 1 & 2 & 1 & -1 \end{array}\right] \overset{R_2 \to R_2 - R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & -1 & -3 & -1 \end{array}\right]$$

$$\overset{R_2 \to -R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right] \overset{R_1 \to R_1 - 3R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 3 & 1 \end{array}\right] \quad (2)$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 4 & 2 & -6 \end{array}\right] \overset{R_2 \to R_2/2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 1 & 2 & 1 & -3 \end{array}\right] \overset{R_2 \to R_2 - R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & -1 & -3 & -3 \end{array}\right]$$

$$\overset{R_2 \to -R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 1 & 3 & 3 \end{array}\right] \overset{R_1 \to R_1 - 3R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -9 \\ 0 & 1 & 3 & 3 \end{array}\right] \quad (3)$$

We then need

$$\begin{array}{rcrcl} c_{11} & & - & 5c_{13} & = 1 \\ & c_{12} & + & 3c_{13} & = 0 \end{array} \quad \text{or} \quad \begin{array}{rcrcr} c_{11} & = & 1 & + & 5c_{13} \\ c_{12} & = & & - & 3c_{13} \\ c_{13} & = & & & c_{13} \end{array} \quad (1)$$

Page 70 Number 44 (continued 3)

Solution (continued).

$$\begin{array}{rcl} c_{21} - 5c_{23} &=& -3 \\ c_{22} + 3c_{23} &=& 1 \end{array} \quad \text{or} \quad \begin{array}{rcl} c_{21} &=& -3 + 5c_{23} \\ c_{22} &=& 1 - 3c_{23} \\ c_{23} &=& c_{23} \end{array} \qquad (2)$$

$$\begin{array}{rcl} c_{31} - 5c_{33} &=& -9 \\ c_{32} + 3c_{33} &=& 3 \end{array} \quad \text{or} \quad \begin{array}{rcl} c_{31} &=& -9 + 5c_{33} \\ c_{32} &=& 3 - 3c_{33} \\ c_{33} &=& c_{33} \end{array} \qquad (3)$$

So in our general solution, c_{13}, c_{23}, and c_{33} act as free variables. To find matrix C, we can therefore pick *any* values for c_{13}, c_{23}, and c_{33} (so there are infinitely many possible choices for C). The easiest choice is to set $c_{13} = c_{23} = c_{33} = 0$ and then we have $c_{11} = 1$, $c_{12} = 0$, $c_{21} = -3$, $c_{22} = 1$, $c_{31} = -9$, and $c_{32} = 3$. This gives $\boxed{C = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -9 & 3 & 0 \end{bmatrix}.}$ □

Page 70 Number 50

Page 70 Number 50. Let A be a 4×4 matrix. Find a matrix C such that the result of applying the sequence of elementary operations:
(1) Interchange Row 1 and Row 4,
(2) Add 6 times Row 2 to Row 1,
(3) Add -3 times Row 1 to Row 3,
(4) Add -2 times Row 4 to Row 2,
to A can also be found by computing the product CA.

Solution. We find the elementary matrices which represent the row operations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{\xrightarrow{R_1 \leftrightarrow R_4}} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = E_1,$$

Page 70 Number 50 (continued 1)

Solution (continued).

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{\xrightarrow{R_1 \to R_1 + 6R_2}} \begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E_2,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{\xrightarrow{R_3 \to R_3 - 3R_1}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E_3,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{\xrightarrow{R_2 \to R_2 - 2R_4}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E_4.$$

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Solution (continued). Then we take $C = E_4E_3E_2E_1$:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 6 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 0 & -18 & 1 & -3 \\ 1 & 0 & 0 & 0 \end{bmatrix}.}$$

□

Page 71 Number 56

Page 71 Number 56. Find a, b, and c such that the parabola $y = ax^2 + bx + c$ passes through the points $(1, -4)$, $(-1, 0)$, and $(2, 3)$.

Solution. We need $(-4) = a(1)^2 + b(1) + c$, $(0) = a(-1)^2 + b(-1) + c$, and $(3) = a(2)^2 + b(2) + c$. So we have the system of equations

$$\begin{aligned} a + b + c &= -4 \\ a - b + c &= 0 \\ 4a + 2b + c &= 3 \end{aligned}$$

so we consider the augmented matrix and reduce it:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 1 & -1 & 1 & 0 \\ 4 & 2 & 1 & 3 \end{array}\right] \underset{R_3 \to R_3 - 4R_1}{\underset{\sim}{R_2 \to R_2 - R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 0 & -2 & 0 & 4 \\ 0 & -2 & -3 & 19 \end{array}\right]$$

$$\underset{\sim}{R_2 \to R_2/(-2)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & -3 & 19 \end{array}\right] \underset{R_3 \to R_3 + 2R_2}{\underset{\sim}{R_1 \to R_1 - R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -3 & 15 \end{array}\right]$$

Page 71 Number 56 (continued)

Page 71 Number 56. Find a, b, and c such that the parabola $y = ax^2 + bx + c$ passes through the points $(1, -4)$, $(-1, 0)$, and $(2, 3)$.

Solution (continued).

$$\underset{\sim}{R_3 \to R_3/(-3)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right] \underset{\sim}{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right].$$

So we take $a = 3$, $b = -2$, $c = -5$ and get the parabola $\boxed{y = 3x^2 - 2x - 5.}$ Note: Just as two distinct points in the plane determine a line, three non-collinear points in a plane determine a parabola. □