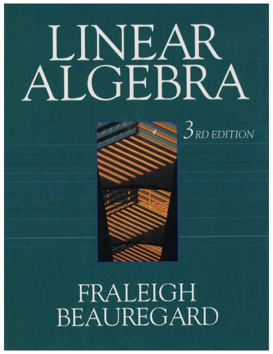
Linear Algebra

Chapter 1. Vectors, Matrices, and Linear Systems

Section 1.5. Inverses of Square Matrices—Proofs of Theorems



Example 1.5.A

Example 1.5.A. It is easy to invert an elementary matrix. For example, suppose E_1 interchanges Row 1 and Row 2 of a 3×3 matrix. Suppose E_2 multiplies Row 2 by 7 in a 3×3 matrix. Find the inverses of E_1 and E_2.

Solution. We have $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To invert the operation of interchanging Row 1 and Row 3 we simply interchange them again. To invert the operation of multiplying Row 2 by 7 we divide Row 2 by 7. So we expect $E_1^{-1} = E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We can easily verify that $E_1 E_1^{-1} = \mathcal{I}$ and $E_2 E_2^{-1} = \mathcal{I}$. □

Lemma 1.1

Lemma 1.1. Condition for $A\vec{x} = \vec{b}$ to be Solvable for $\vec{b}$.
Let A be an $n \times n$ matrix. The linear system $A\vec{x} = \vec{b}$ has a solution for every choice of column vector $\vec{b} \in \mathbb{R}^n$ if and only if A is row equivalent to the $n \times n$ identity matrix $\mathcal{I}$.

Proof. Suppose A is row equivalent to $\mathcal{I}$. Let $\vec{b}$ by any column vector in $\mathbb{R}^n$. Then $[A \mid \vec{b}] \sim [\mathcal{I} \mid \vec{c}]$ for some column vector $\vec{c} \in \mathbb{R}^n$. Then, by Theorem 1.6, $\vec{x} = \vec{c}$ is a solution to $A\vec{x} = \vec{b}$.

Suppose A is not row equivalent to $\mathcal{I}$. Row reduce A to a reduced row echelon form H (so $H \neq \mathcal{I}$). So the last row (i.e., the nth row) of H must be all zeros. Now the row reduction of A to H can be accomplished by multiplication on the left by a sequence of elementary matrices by repeated application of Theorem 1.8, "Use of Elementary Matrices." Say $E_t \cdots E_2 E_1 A = H$. Now elementary matrices are invertible (see Example 1.5.A). Let $\vec{e}_n$ be the nth basis element of $\mathbb{R}^n$ written as a column vector. Define $\vec{b} = (E_t \cdots E_2 E_1)^{-1} \vec{e}_n$.

Lemma 1.1 (continued)

Lemma 1.1. Condition for $A\vec{x} = \vec{b}$ to be Solvable for $\vec{b}$.
Let A be an $n \times n$ matrix. The linear system $A\vec{x} = \vec{b}$ has a solution for every choice of column vector $\vec{b} \in \mathbb{R}^n$ if and only if A is row equivalent to the $n \times n$ identity matrix $\mathcal{I}$.
Proof (continued). Consider the system of equations $A\vec{x} = \vec{b}$ with associated augmented matrix $[A \mid \vec{b}]$. Applying the sequence of elementary row operations associated with $E_t \cdots E_2 E_1$ reduces $[A \mid \vec{b}]$ to

$$[E_t \cdots E_2 E_1 A \mid E_t \cdots E_2 E_1 \vec{b}] = [E_t \cdots E_2 E_1 A \mid (E_t \cdots E_2 E_1)(E_t \cdots E_2 E_1)^{-1} \vec{e}_n]$$

$$= [H \mid \vec{e}_n].$$

But then the last row of H consists of all zeros to the left of the partition and 1 to the right of the partition. So by Theorem 1.7(1), "Solutions of $A\vec{x} = \vec{b}$," $A\vec{x} = \vec{b}$ has no solution. So if A is not row equivalent to $\mathcal{I}$ then the system $A\vec{x} = \vec{b}$ does not have a solution for *all* $\vec{b} \in \mathbb{R}^n$. □

Page 84 Number 12

Page 84 Number 12. Determine whether the span of the column vectors of $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 5 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ -1 & 2 & 4 & -2 \end{bmatrix}$ span $\mathbb{R}^4$.

Solution. Recall that for any $\vec{x} \in \mathbb{R}^n$, $A\vec{x}$ is a linear combination of the columns of A by Note 1.3.A. So to see if the column vectors of A span $\mathbb{R}^4$, we need to choose an arbitrary $\vec{b} \in \mathbb{R}^4$ and see if there is $\vec{x} \in \mathbb{R}^4$ such that $A\vec{x} = \vec{b}$. That is, we need to see if $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^4$. So by Lemma 1.1 we only need to see if A is row equivalent to $\mathcal{I}$. Consider

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 5 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ -1 & 2 & 4 & -2 \end{bmatrix} \underset{R_4 \to R_4 + R_1}{\overset{R_2 \to R_2 + 3R_1}{\sim}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 5 & -2 \end{bmatrix}$$

Page 84 Number 12 (continued)

Solution (continued).

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 5 & -2 \end{bmatrix} \underset{R_3 \to R_3 + R_2}{\overset{R_1 \to R_1 - 2R_2}{\sim}} \begin{bmatrix} 1 & 0 & -5 & -4 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 5 & -2 \end{bmatrix}$$

$$\underset{R_4 \to R_4 - R_3}{\overset{\substack{R_1 \to R_1 + R_3 \\ R_2 \to R_2 - (3/5)R_3}}{\sim}} \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & -1 & 0 & 16/5 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_3 \to R_3/5}{\overset{R_2 \to -R_2}{\sim}} \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -16/5 \\ 0 & 0 & 1 & -2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.$$

Now H is in reduced row echelon form and $H \neq \mathcal{I}$. So Lemma 1.1 implies that $\boxed{\text{NO, the columns do not span } \mathbb{R}^4.}$ □

Theorem 1.11

Theorem 1.11. A Commutivity Property.
Let A and C be $n \times n$ matrices. Then $CA = \mathcal{I}$ if and only if $AC = \mathcal{I}$.

Proof. Suppose that $AC = \mathcal{I}$. Then the equation $A\vec{x} = \vec{b}$ has a solution for every column vector $\vec{b} \in \mathbb{R}^n$. Notice that $\vec{x} = C\vec{b}$ is a solution because

$$A(C\vec{b}) = (AC)\vec{b} = \mathcal{I}\vec{b} = \vec{b}.$$

By Lemma 1.1, we know that A is row equivalent to the $n \times n$ identity matrix $\mathcal{I}$, and so there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_t$ such that $(E_t \cdots E_2 E_1)A = \mathcal{I}$. By Theorem 1.9, the two equations

$$(E_t \cdots E_2 E_1)A = \mathcal{I} \text{ and } AC = \mathcal{I}$$

imply that $E_t \cdots E_2 E_1 = C$, and so we have $CA = \mathcal{I}$. The other half of the proof follows by interchanging the roles of A and C. □

Page 84 Number 4

Page 84 Number 4. Consider $A = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$. Find A^{-1}. Use A^{-1} to solve the system
$$\begin{aligned} 6x_1 + 7x_2 &= 4 \\ 8x_1 + 9x_2 &= 6. \end{aligned}$$

Solution. We form $[A|\mathcal{I}]$ and apply Gauss-Jordan elimination to produce the row equivalent $[\mathcal{I}|A^{-1}]$ (if possible). So

$$\left[\begin{array}{cc|cc} 6 & 7 & 1 & 0 \\ 8 & 9 & 0 & 1 \end{array}\right] \overset{R_1 \to R_1/6}{\sim} \left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 8 & 9 & 0 & 1 \end{array}\right]$$

$$\overset{R_2 \to R_2 - 8R_1}{\sim} \left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 8 - 8(1) & 9 - 8(7/6) & 0 - 8(1/6) & 1 - 8(0) \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 0 & -1/3 & -4/3 & 1 \end{array}\right] \overset{R_2 \to -3R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 0 & 1 & 4 & -3 \end{array}\right]$$

Page 84 Number 4 (continued 1)

Solution (continued).

$$\left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 0 & 1 & 4 & -3 \end{array}\right] \underset{\sim}{\overset{R_1 \to R_1 - (7/6)R_2}{}}$$

$$\left[\begin{array}{cc|cc} 1-(7/6)(0) & 7/6-(7/6)(1) & 1/6-(7/6)(4) & 0-(7/6)(-3) \\ 0 & 1 & 4 & -3 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & -9/2 & 7/2 \\ 0 & 1 & 4 & -3 \end{array}\right].$$

So $\boxed{A^{-1} = \begin{bmatrix} -9/2 & 7/2 \\ 4 & -3 \end{bmatrix}.}$

Page 84 Number 4 (continued 2)

Solution (continued). For the system of equations, we express it as a matrix product $A\vec{x} = \vec{b}$: $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. Then $A^{-1}A\vec{x} = A^{-1}\vec{b}$ or $\mathcal{I}\vec{x} = A^{-1}\vec{b}$ or $\vec{x} = A^{-1}\vec{b}$. So

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\vec{b} = \begin{bmatrix} -9/2 & 7/2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} (-9/2)(4) + (7/2)(6) \\ 4(4) - 3(6) \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

and the solution is $\boxed{x_1 = 3,\ x_2 = -2.}$ □

Page 85 Number 24

Page 85 number 24. Prove that if A is an invertible $n \times n$ matrix then A^T is invertible. Describe $(A^T)^{-1}$ in terms of A^{-1}.

Solution. We know that $(AB)^T = B^T A^T$ (see "Properties of the Transpose Operator" in Section 1.3; page 4 of the notes). Since A is invertible then $AA^{-1} = A^{-1}A = \mathcal{I}$. So $(AA^{-1})^T = (A^{-1}A)^T = \mathcal{I}^T = \mathcal{I}$ (since the identity matrix $\mathcal{I}$ is symmetric; see Definition 1.11). Hence $(A^{-1})^T A^T = A^T (A^{-1})^T = \mathcal{I}$ and so the inverse of A^T is $(A^{-1})^T$. Therefore A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$. □

Page 86 Number 30

Page 86 number 30. A square matrix A is said to be *idempotent* if $A^2 = A$.

(a) Give an example of an idempotent matrix other than 0 and $\mathcal{I}$.

Solution. An easy example can be found by slightly modifying $\mathcal{I}$. Consider, say, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A.$$

(b) Prove that if matrix A is both idempotent and invertible, then $A = \mathcal{I}$.

Proof. Suppose $A^2 = A$ and A^{-1} exists. Then $A^{-1}(A^2) = A^{-1}A$ and by associativity (Theorem 1.3.A(8)), "Properties of Matrix Algebra") $(A^{-1}A)A = A^{-1}A$ or $\mathcal{I}A = \mathcal{I}$ or $A = \mathcal{I}$. □