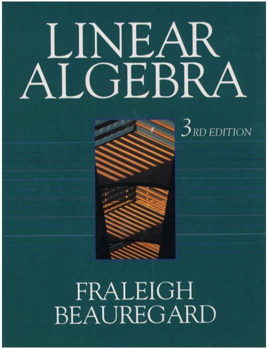
Linear Algebra

Chapter 1. Vectors, Matrices, and Linear Systems

Section 1.5. Inverses of Square Matrices—Proofs of Theorems



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Example 1.5.A. It is easy to invert an elementary matrix. For example, suppose E_1 interchanges Row 1 and Row 2 of a 3×3 matrix. Suppose E_2 multiplies Row 2 by 7 in a 3×3 matrix. Find the inverses of E_1 and E_2.

Solution. We have $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

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Lemma 1.1

Lemma 1.1. Condition for $A\vec{x} = \vec{b}$ to be Solvable for $\vec{b}$.
Let A be an $n \times n$ matrix. The linear system $A\vec{x} = \vec{b}$ has a solution for every choice of column vector $\vec{b} \in \mathbb{R}^n$ if and only if A is row equivalent to the $n \times n$ identity matrix $\mathcal{I}$.

Proof. Suppose A is row equivalent to $\mathcal{I}$. Let $\vec{b}$ by any column vector in $\mathbb{R}^n$. Then $[A \mid \vec{b}] \sim [\mathcal{I} \mid \vec{c}]$ for some column vector $\vec{c} \in \mathbb{R}^n$. Then, by Theorem 1.6, $\vec{x} = \vec{c}$ is a solution to $A\vec{x} = \vec{b}$.

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Proof (continued). Consider the system of equations $A\vec{x} = \vec{b}$ with associated augmented matrix $[A \mid \vec{b}]$. Applying the sequence of elementary row operations associated with $E_t \cdots E_2 E_1$ reduces $[A \mid \vec{b}]$ to

$$[E_t \cdots E_2 E_1 A \mid E_t \cdots E_2 E_1 \vec{b}] = [E_t \cdots E_2 E_1 A \mid (E_t \cdots E_2 E_1)(E_t \cdots E_2 E_1)^{-1} \vec{e}_n]$$

$$= [H \mid \vec{e}_n].$$

But then the last row of H consists of all zeros to the left of the partition and 1 to the right of the partition. So by Theorem 1.7(1), "Solutions of $A\vec{x} = \vec{b}$," $A\vec{x} = \vec{b}$ has no solution. So if A is not row equivalent to $\mathcal{I}$ then the system $A\vec{x} = \vec{b}$ does not have a solution for *all* $\vec{b} \in \mathbb{R}^n$. □

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Page 84 Number 12. Determine whether the span of the column vectors of $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 5 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ -1 & 2 & 4 & -2 \end{bmatrix}$ span $\mathbb{R}^4$.

Solution. Recall that for any $\vec{x} \in \mathbb{R}^n$, $A\vec{x}$ is a linear combination of the columns of A by Note 1.3.A. So to see if the column vectors of A span $\mathbb{R}^4$, we need to choose an arbitrary $\vec{b} \in \mathbb{R}^4$ and see if there is $\vec{x} \in \mathbb{R}^4$ such that $A\vec{x} = \vec{b}$. That is, we need to see if $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^4$.

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$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 5 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ -1 & 2 & 4 & -2 \end{bmatrix} \underset{R_4 \to R_4 + R_1}{\overset{R_2 \to R_2 + 3R_1}{\widetilde{\qquad}}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 5 & -2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 5 & -2 \end{bmatrix} \underbrace{\overset{R_1 \to R_1 - 2R_2}{\sim}}_{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & -5 & -4 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 5 & -2 \end{bmatrix}$$

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Theorem 1.11

Theorem 1.11. A Commutivity Property.
Let A and C be $n \times n$ matrices. Then $CA = \mathcal{I}$ if and only if $AC = \mathcal{I}$.

Proof. Suppose that $AC = \mathcal{I}$. Then the equation $A\vec{x} = \vec{b}$ has a solution for every column vector $\vec{b} \in \mathbb{R}^n$. Notice that $\vec{x} = C\vec{b}$ is a solution because

$$A(C\vec{b}) = (AC)\vec{b} = \mathcal{I}\vec{b} = \vec{b}.$$

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$$(E_t \cdots E_2 E_1)A = \mathcal{I} \text{ and } AC = \mathcal{I}$$

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$$\begin{array}{rcl} 6x_1 + 7x_2 & = & 4 \\ 8x_1 + 9x_2 & = & 6. \end{array}$$

Solution. We form $[A|\mathcal{I}]$ and apply Gauss-Jordan elimination to produce the row equivalent $[\mathcal{I}|A^{-1}]$ (if possible).

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$$\left[\begin{array}{cc|cc} 6 & 7 & 1 & 0 \\ 8 & 9 & 0 & 1 \end{array}\right] \overset{R_1 \to R_1/6}{\widetilde{\qquad}} \left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 8 & 9 & 0 & 1 \end{array}\right]$$

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$$\overset{R_2 \to R_2 - 8R_1}{\sim} \left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 8-8(1) & 9-8(7/6) & 0-8(1/6) & 1-8(0) \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 0 & -1/3 & -4/3 & 1 \end{array}\right] \overset{R_2 \to -3R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 0 & 1 & 4 & -3 \end{array}\right]$$

Page 84 Number 4 (continued 1)

Solution (continued).

$$\left[\begin{array}{cc|cc} 1 & 7/6 & 1/6 & 0 \\ 0 & 1 & 4 & -3 \end{array}\right] \overset{R_1 \to R_1 - (7/6)R_2}{\sim}$$

$$\left[\begin{array}{cc|cc} 1-(7/6)(0) & 7/6-(7/6)(1) & 1/6-(7/6)(4) & 0-(7/6)(-3) \\ 0 & 1 & 4 & -3 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & -9/2 & 7/2 \\ 0 & 1 & 4 & -3 \end{array}\right].$$

So $\boxed{A^{-1} = \begin{bmatrix} -9/2 & 7/2 \\ 4 & -3 \end{bmatrix}}.$

Page 84 Number 4 (continued 1)

Solution (continued).

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Page 84 Number 4 (continued 2)

Solution (continued). For the system of equations, we express it as a matrix product $A\vec{x} = \vec{b}$: $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. Then $A^{-1}A\vec{x} = A^{-1}\vec{b}$ or $\mathcal{I}\vec{x} = A^{-1}\vec{b}$ or $\vec{x} = A^{-1}\vec{b}$. So

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and the solution is $\boxed{x_1 = 3,\ x_2 = -2.}$ □

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Page 85 Number 24

Page 85 number 24. Prove that if A is an invertible $n \times n$ matrix then A^T is invertible. Describe $(A^T)^{-1}$ in terms of A^{-1}.

Solution. We know that $(AB)^T = B^T A^T$ (see "Properties of the Transpose Operator" in Section 1.3; page 4 of the notes).

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Solution. We know that $(AB)^T = B^T A^T$ (see "Properties of the Transpose Operator" in Section 1.3; page 4 of the notes). Since A is invertible then $AA^{-1} = A^{-1}A = \mathcal{I}$. So $(AA^{-1})^T = (A^{-1}A)^T = \mathcal{I}^T = \mathcal{I}$ (since the identity matrix $\mathcal{I}$ is symmetric; see Definition 1.11).

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Page 86 Number 30

Page 86 number 30. A square matrix A is said to be *idempotent* if $A^2 = A$.

(a) Give an example of an idempotent matrix other than 0 and $\mathcal{I}$.

Solution. An easy example can be found by slightly modifying $\mathcal{I}$.

Consider, say, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A.$$

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(b) Prove that if matrix A is both idempotent and invertible, then $A = \mathcal{I}$.

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Proof. Suppose $A^2 = A$ and A^{-1} exists. Then $A^{-1}(A^2) = A^{-1}A$ and by associativity (Theorem 1.3.A(8)), "Properties of Matrix Algebra") $(A^{-1}A)A = A^{-1}A$ or $\mathcal{I}A = \mathcal{I}$ or $A = \mathcal{I}$. □

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