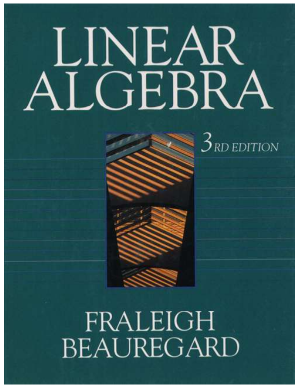
# Linear Algebra

**Chapter 1. Vectors, Matrices, and Linear Systems**
Section 1.6. Homogeneous Systems, Subspaces, and Bases—Proofs of Theorems








## Page 99 Number 8

**Page 99 Number 8.** Determine whether the set $W = \{[2x, x+y, y] \mid x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

**Solution.** By Definition 1.16, we need to see if $W$ is closed under vector addition and scalar multiplication. Let $\vec{v}_1, \vec{v}_2 \in W$. Then $\vec{v}_1 = [2x_1, x_1 + y_1, y_1]$ and $\vec{v}_2 = [2x_2, x_2 + y_2, y_2]$ for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$. For vector addition,

$$\begin{aligned}
\vec{v}_1 + \vec{v}_2 &= [2x_1, x_1 + y_1, y_1] + [2x_2, x_2 + y_2, y_2] \\
&= [2x_1 + 2x_2, (x_1 + y_1) + (x_2 + y_2), y_1 + y_2] \\
&= [2(x_1 + x_2), (x_1 + x_2) + (y_1 + y_2), (y_1 + y_2)] \\
&= [2x, x + y, y] \text{ where } x = x_1 + x_2 \text{ and } y = y_1 + y_2.
\end{aligned}$$

So $\vec{v}_1 + \vec{v}_2 \in W$ and $W$ is closed under vector addition.





## Page 99 Number 8 (continued)

**Page 99 Number 8.** Determine whether the set $W = \{[2x, x+y, y] \mid x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

**Solution (continued).** For scalar multiplication, let $r \in \mathbb{R}$ and consider

$$\begin{aligned}
r\vec{v}_1 &= r[2x_1, x_1 + y_1, y_1] = [r(2x_1), r(x_1 + y_1), r(y_1)] \\
&= [2(rx_1), (rx_1) + (ry_1), (ry_1)] \\
&= [2x, x + y, y] \text{ where } x = rx_1 \text{ and } y = ry_1.
\end{aligned}$$

So $r\vec{v}_1 \in W$ and $W$ is closed under scalar multiplication. Therefore, $W$ is a subspace of $\mathbb{R}^3$. $\square$





## Theorem 1.14

**Theorem 1.14. Subspace Property of a Span**
Let $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ be the span of $k > 0$ vectors in $\mathbb{R}^n$ Then $W$ is a subspace of $\mathbb{R}^n$. (The vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are said to *span* or *generate* the subspace.)

**Proof.** We use Definition 1.16, "Closure and Subspace." Let $\vec{u}, \vec{v} \in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and let $c$ be a scalar. Then $\vec{u} = r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k$ and $\vec{v} = s_1\vec{w}_1 + s_2\vec{w}_2 + \cdots + s_k\vec{w}_k$ for some scalars $r_i$ and $s_i$. Then

$$\begin{aligned}
\vec{u} + \vec{v} &= (r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k) + (s_1\vec{w}_1 + s_2\vec{w}_2 + \cdots + s_k\vec{w}_k) \\
&= (r_1 + s_1)\vec{w}_1 + (r_2 + s_2)\vec{w}_2 + \cdots + (r_k + s_k)\vec{w}_k \text{ by S1 and S2} \\
&\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)
\end{aligned}$$

and so $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is closed under vector addition.

## Theorem 1.14 (continued)

**Theorem 1.14. Subspace Property of a Span**
Let $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ be the span of $k > 0$ vectors in $\mathbb{R}^n$ Then $W$ is a subspace of $\mathbb{R}^n$. (The vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are said to *span* or *generate* the subspace.)

**Proof (continued).** Next,

$$\begin{aligned} c\vec{u} &= c(r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k) \\ &= (cr_1)\vec{w}_1 + (cr_2)\vec{w}_2 + \cdots + (cr_k)\vec{w}_k \text{ by S1 and S3} \\ &\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \end{aligned}$$

and so $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is closed under scalar multiplication. So by Definition 1.16 $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is a subspace of $\mathbb{R}^n$. □





## Page 100 Number 18

**Page 100 Number 18.** Find a generating set for the solution set of the homogeneous linear system:

$$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & x_3 & - & x_4 & = & 0 \\ & & x_2 & + & x_3 & & & = & 0 \\ x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 0. \end{array}$$

**Solution.** We apply Gauss-Jordan elimination to the augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 3 & 0 \end{array}\right] \underrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & 4 & 0 \end{array}\right]$$

$$\underrightarrow{\substack{R_1 \to R_1 + R_2 \\ R_3 \to R_3 - 3R_2}} \left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & 4 & 0 \end{array}\right] \underrightarrow{R_3 \to R_3/(-5)} \left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4/5 & 0 \end{array}\right]$$

...





## Page 100 Number 18 (continued 1)

**Solution (continued).** ...

$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4/5 & 0 \end{array}\right] \underrightarrow{\substack{R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 - R_3}} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 3/5 & 0 \\ 0 & 1 & 0 & 4/5 & 0 \\ 0 & 0 & 1 & -4/5 & 0 \end{array}\right].$$

Returning to a system of equations,

$$\begin{array}{lcl} x_1 + (3/5)x_4 = 0 & \text{or} & x_1 = -(3/5)x_4 \\ x_2 + (4/5)x_4 = 0 & & x_2 = -(4/5)x_4 \\ x_3 - (4/5)x_4 = 0 & & x_3 = (4/5)x_4 \\ & & x_4 = x_4. \end{array}$$

So let $r = x_4$ be a free variable and we have that the general solution is of the form $\vec{x} \in \left\{ r \begin{bmatrix} -3/5 \\ -4/5 \\ 4/5 \\ 1 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$.





## Page 100 Number 18 (continued 2)

**Solution (continued).** So a generating set for the system is

$$\boxed{\left\{ \begin{bmatrix} -3/5 \\ -4/5 \\ 4/5 \\ 1 \end{bmatrix} \right\}.}$$

Note: We could have let $s = x_4/5$ be a free variable in which case a generating set is given by the simpler $\left\{ \begin{bmatrix} -3 \\ -4 \\ 4 \\ 5 \end{bmatrix} \right\}$. □

# Theorem 1.15

**Theorem 1.15. Unique Linear Combinations.**
The set $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ if and only if

$$r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$$

implies

$$r_1 = r_2 = \cdots = r_k = 0.$$

**Proof.** Suppose $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$. Then by Definition 1.17, "Basis for a Subspace," every vector in $W$ can be expressed uniquely as a linear combination of the $\vec{w}_i$. In particular, $r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$ is satisfied for $r_1 = r_2 = \cdots = r_k = 0$ and the uniqueness condition implies that we must have $r_1 = r_2 = \cdots = r_k = 0$.





# Theorem 1.15 (continued)

**Theorem 1.15. Unique Linear Combinations.**
The set $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ if and only if

$$r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$$

implies

$$r_1 = r_2 = \cdots = r_k = 0.$$

**Proof (continued).** Now suppose that $r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$ implies that $r_1 = r_2 = \cdots = r_k = 0$. Let $\vec{w} \in W$ and suppose $\vec{w} = c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_k\vec{w}_k = d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_k\vec{w}_k$. Then $\vec{0} = \vec{w} - \vec{w} = (c_1 - d_1)\vec{w}_1 + (c_2 - d_2)\vec{w}_2 + \cdots + (c_k - d_k)\vec{w}_k$ (by S1 and S2). By hypothesis for this case, we must have $c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0$. That is, we must have $c_1 = d_1$, $c_2 = d_2$, ..., $c_k = d_k$. Hence every vector of $W$ is a unique linear combination of the $\vec{w}_i$, as claimed. □





# Page 100 Number 22(a)

**Page 100 Number 22(a).** Use Theorem 1.15 to determine whether the set $\{[-1,1],[1,2]\}$ is a basis for the subspace of $\mathbb{R}^2$ that it spans.

**Solution.** Based on Theorem 1.15, we consider scalars $r_1, r_2 \in \mathbb{R}$ such that $r_1[-1,1] + r_2[1,2] = [0,0]$. This implies $[-r_1, r_1] + [r_2, 2r_2] = [0,0]$ or $[-r_1 + r_2, r_1 + 2r_2] = [0,0]$. So we need

$$\begin{aligned} -r_1 + r_2 &= 0 \qquad (1) \\ r_1 + 2r_2 &= 0. \qquad (2) \end{aligned}$$

From (1) we see that $r_1 = r_2$ and so from (2) we need $r_1 + 2(r_1) = 0$ or $3r_1 = 0$ or $r_1 = 0$. Since $r_1 = r_2$ we also need $r_2 = 0$. Hence we must have $r_1 = r_2 = 0$ and so $\{[-1,1],[1,2]\}$ is a basis for its span by Theorem 1.15. □





# Page 100 Number 22(b)

**Page 100 Number 22(b).** Use Theorem 1.16 to determine whether the set $\{[-1,1],[1,2]\}$ is a basis for the subspace of $\mathbb{R}^2$ that it spans.

**Solution.** We define matrix $A$ which has as its *columns* the vectors in the set: $A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$. By Theorem 1.16, we see that the columns of $A$ form a basis for $\mathbb{R}^2$ if and only if $A$ is row equivalent to $\mathcal{I}$. So we row reduce $A$:

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \overset{R_2 \to R_2 + R_1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$\overset{R_2 \to R_2/3}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \overset{R_1 \to R_1 - 2R_2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{I}.$$

So $A \sim \mathcal{I}$ and hence the columns of $A$ form a basis for $\mathbb{R}^2$; that is, the set $\{[-1,1],[1,2]\}$ is a basis for the subspace of $\mathbb{R}^2$ that it spans. (Since there are two vectors, their span is all of $\mathbb{R}^2$.) □

## Page 97 Example 6

**Example.** Page 97 Example 6. A basis of $\mathbb{R}^n$ cannot contain more than $n$ vectors.

**Proof.** Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for $\mathbb{R}^n$ and ASSUME $k > n$. Consider the system $A\vec{x} = \vec{0}$ where the column vectors of $A$ are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. Then $A$ has $n$ rows and $k$ columns (corresponding to $n$ equations in $k$ unknowns). With $n < k$, Corollary 2 implies there is a nontrivial solution to $A\vec{x} = \vec{0}$. But this corresponds to a linear combination of the columns of $A$ which equals $\vec{0}$ while not all the coefficients are 0. This CONTRADICTS Theorem 1.15 (since we then have two different linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ which equal $\vec{0}$). So the assumption that $k > n$ is false. Therefore, $k \le n$. □





## Theorem 1.18

**Theorem 1.18. Structure of the Solution Set of $A\vec{x} = \vec{b}$.**
Let $A\vec{x} = \vec{b}$ be a linear system. If $\vec{p}$ is any particular solution of $A\vec{x} = \vec{b}$ and $\vec{h}$ is a solution to $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$. In fact, every solution of $A\vec{x} = \vec{b}$ has the form $\vec{p} + \vec{h}$ and the general solution is $\vec{x} = \vec{p} + \vec{h}$ where $A\vec{h} = \vec{0}$ (that is, $\vec{h}$ is an arbitrary element of the nullspace of $A$).

**Proof.** Let $\vec{p}$ be a particular solution of $A\vec{x} = \vec{b}$ (so that $A\vec{p} = \vec{b}$). Let $\vec{h}$ be a solution of the homogeneous system $A\vec{x} = \vec{0}$ (so that $A\vec{h} = \vec{0}$). Then

$$\begin{aligned} A(\vec{p} + \vec{h}) &= A\vec{p} + A\vec{h} \text{ by Theorem 1.2.A(10),} \\ & \quad \text{Distribution of Matrix Multiplication} \\ &= \vec{b} + \vec{0} = \vec{b}. \end{aligned}$$

So $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$.





## Theorem 1.18 (continued)

**Theorem 1.18. Structure of the Solution Set of $A\vec{x} = \vec{b}$.**
Let $A\vec{x} = \vec{b}$ be a linear system. If $\vec{p}$ is any particular solution of $A\vec{x} = \vec{b}$ and $\vec{h}$ is a solution to $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$. In fact, every solution of $A\vec{x} = \vec{b}$ has the form $\vec{p} + \vec{h}$ and the general solution is $\vec{x} = \vec{p} + \vec{h}$ where $A\vec{h} = \vec{0}$ (that is, $\vec{h}$ is an arbitrary element of the nullspace of $A$).

**Proof (continued).** Now suppose $\vec{q}$ is any solution to $A\vec{x} = \vec{b}$. With $\vec{p}$ as a particular solution to $A\vec{x} = \vec{b}$ we have

$$\begin{aligned} A(\vec{q} - \vec{p}) &= A\vec{q} - A\vec{p} \text{ by Theorem 1.2.A(10),} \\ & \quad \text{Distribution of Matrix Multiplication} \\ &= \vec{b} - \vec{b} = \vec{0}. \end{aligned}$$

So $\vec{q} - \vec{p}$ is a solution of $A\vec{x} = \vec{0}$, say $\vec{q} - \vec{p} = \vec{h}$. So $\vec{q} = \vec{p} + \vec{h}$ and every solution $\vec{x}$ of $A\vec{x} = \vec{b}$ is of the form $\vec{p} + \vec{h}$ where $\vec{p}$ is a particular solution of $A\vec{x} + \vec{b}$ and $\vec{h}$ is any solution of $A\vec{x} = \vec{h}$. □





## Page 100 Number 36

**Page 100 Number 36.** Solve the linear system

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & 4 \\ 2x_1 & + & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ x_1 & - & 7x_2 & - & 6x_3 & + & 2x_4 & = & 6 \end{array}$$

and express the solution set in a form that illustrates Theorem 1.18.

**Solution.** We apply Gauss-Jordan elimination to the augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 4 \\ 2 & 1 & -3 & -1 & 6 \\ 1 & -7 & -6 & 2 & 6 \end{array}\right] \underset{R_3 \to R_3 - R_1}{\underbrace{R_2 \to R_2 - 2R_1}} \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 4 \\ 0 & 5 & -5 & -3 & -2 \\ 0 & -5 & -7 & 1 & 2 \end{array}\right] \underbrace{R_3 \to R_3 + R_2}$$

$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 4 \\ 0 & 5 & -5 & -3 & -2 \\ 0 & 0 & -12 & -2 & 0 \end{array}\right] \underset{R_3 \to R_3/(-12)}{\underbrace{R_2 \to R_2/5}} \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 4 \\ 0 & 1 & -1 & -3/5 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right]$$

## Page 100 Number 36 (continued 1)

**Solution (continued).**

$$\underset{R_1 \to R_1 + 2R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1/5 & 16/5 \\ 0 & 1 & -1 & -3/5 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right]$$

$$\underset{\substack{R_1 \to R_1 + R_3 \\ R_2 \to R_2 + R_3}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/30 & 16/5 \\ 0 & 1 & 0 & -13/30 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right].$$

This corresponds to the system of equations:

$$\begin{array}{ccccccc} x_1 & & & - & (1/30)x_4 & = & 16/5 \\ & x_2 & & - & (13/30)x_4 & = & -2/5 \\ & & x_3 & + & (1/6)x_4 & = & 0 \end{array}$$

For a particular solution $\vec{p}$ to the original system of equations we choose to set $x_4 = 0$ so that...





## Page 100 Number 36 (continued 2)

**Solution (continued).** $\ldots \vec{p} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix}$. With $A$ as the coefficient matrix, the homogeneous system $A\vec{x} = \vec{0}$ reduces to a similar system of equations but with only 0's on the right hand side:

$$\begin{array}{ccccccc} x_1 & & & - & (1/30)x_4 & = & 0 \\ & x_2 & & - & (13/30)x_4 & = & 0 \\ & & x_3 & + & (1/6)x_4 & = & 0 \end{array} \quad \text{or} \quad \begin{array}{l} x_1 = (1/30)x_4 \\ x_2 = (13/30)x_4 \\ x_3 = -(1/6)x_4 \\ x_4 = x_4 \end{array}$$

So $x_4$ acts as a free variable in the associated homogeneous system of equations. To simplify the numbers, we set $x_4 = 30r$ where $r \in \mathbb{R}$ (since $r$ is any element of $\mathbb{R}$ then $30r$ is any element of $\mathbb{R}$, and conversely).





## Page 100 Number 36 (continued 3)

**Solution (continued).** This gives the solution to the homogeneous system as $\begin{array}{l} x_1 = (1/30)(30r) = r \\ x_2 = (13/30)(30r) = 13r \\ x_3 = -(1/6)(30r) = -5r \\ x_4 = 30r. \end{array}$ So the solution *set* to the homogeneous system of equations $A\vec{x} = \vec{b}$ is $\left\{ r \begin{bmatrix} 1 \\ 13 \\ -5 \\ 30 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$ (this is the nullspace of $A$). Therefore, in the notation of Theorem 1.18, the general solutions to the original (nonhomogeneous) system of equations is

$$\boxed{\vec{x} = \vec{p} + \vec{h} \text{ where } \vec{p} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{h} \in \left\{ r \begin{bmatrix} 1 \\ 13 \\ -5 \\ 30 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}.} \quad \square$$





## Page 101 Number 43

**Page 101 Number 43.** Use Theorem 1.18 to prove why no system of linear equations can have exactly two solutions.

**Proof.** ASSUME to the contrary that linear system $A\vec{x} = \vec{b}$ does have exactly two solutions, say $\vec{p}_1$ and $\vec{p}_2$ (where $\vec{p}_1 \neq \vec{p}_2$). Then $A(\vec{p}_1 - \vec{p}_2) = A\vec{p}_1 - A\vec{p}_2 = \vec{b} - \vec{b} = \vec{0}$ and so $\vec{h} = \vec{p}_1 - \vec{p}_2 \neq 0$ is a solution to the homogeneous system $A\vec{x} = \vec{0}$. Now $\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_1$ (since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_1$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction) and $\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_2$ (since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_2$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction). So by Theorem 1.18, $\vec{p}_1 + \vec{h}$ is a third solution to $A\vec{x} = \vec{b}$. This is a CONTRADICTION to the hypotheses. So the assumption that $A\vec{x} = \vec{b}$ has exactly two solutions is false and the claim follows. $\square$

# Page 101 Number 47

**Page 101 Number 47.** Let $W_1$ and $W_2$ be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^n$.

**Proof.** We use Definition 1.16, "Closure and Subspace." Let $\vec{u}, \vec{v} \in W_1 \cap W_2$. Since $W_1$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_1$. Since $W_2$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_2$. Hence $\vec{u} + \vec{v}$ is in both $W_1$ and $W_2$; that is, $\vec{u} + \vec{v} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under vector addition.

Now let $r$ be a scalar. Since $W_1$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_1$. Since $W_2$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_2$. Hence $r\vec{u}$ is in both $W_1$ and $W_2$; that is, $r\vec{u} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under scalar multiplication. By Definition 1.16, $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$ □