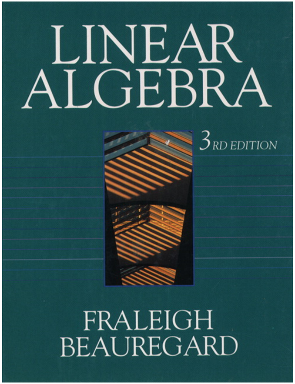
Linear Algebra

Chapter 1. Vectors, Matrices, and Linear Systems

Section 1.6. Homogeneous Systems, Subspaces, and Bases—Proofs of Theorems



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Page 99 Number 8. Determine whether the set $W = \{[2x, x+y, y] \mid x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Solution. By Definition 1.16, we need to see if W is closed under vector addition and scalar multiplication. Let $\vec{v}_1, \vec{v}_2 \in W$. Then $\vec{v}_1 = [2x_1, x_1 + y_1, y_1]$ and $\vec{v}_2 = [2x_2, x_2 + y_2, y_2]$ for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

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$$\begin{aligned}
\vec{v}_1 + \vec{v}_2 &= [2x_1, x_1 + y_1, y_1] + [2x_2, x_2 + y_2, y_2] \\
&= [2x_1 + 2x_2, (x_1 + y_1) + (x_2 + y_2), y_1 + y_2] \\
&= [2(x_1 + x_2), (x_1 + x_2) + (y_1 + y_2), (y_1 + y_2)] \\
&= [2x, x + y, y] \text{ where } x = x_1 + x_2 \text{ and } y = y_1 + y_2.
\end{aligned}$$

So $\vec{v}_1 + \vec{v}_2 \in W$ and W is closed under vector addition.

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$$\begin{aligned} \vec{v}_1 + \vec{v}_2 &= [2x_1, x_1 + y_1, y_1] + [2x_2, x_2 + y_2, y_2] \\ &= [2x_1 + 2x_2, (x_1 + y_1) + (x_2 + y_2), y_1 + y_2] \\ &= [2(x_1 + x_2), (x_1 + x_2) + (y_1 + y_2), (y_1 + y_2)] \\ &= [2x, x + y, y] \text{ where } x = x_1 + x_2 \text{ and } y = y_1 + y_2. \end{aligned}$$

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Solution (continued). For scalar multiplication, let $r \in \mathbb{R}$ and consider

$$\begin{aligned} r\vec{v}_1 &= r[2x_1, x_1 + y_1, y_1] = [r(2x_1), r(x_1 + y_1), r(y_1)] \\ &= [2(rx_1), (rx_1) + (ry_1), (ry_1)] \\ &= [2x, x + y, y] \text{ where } x = rx_1 \text{ and } y = ry_1. \end{aligned}$$

So $r\vec{v}_1 \in W$ and W is closed under scalar multiplication. Therefore, W is a subspace of $\mathbb{R}^3$. □

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So $r\vec{v}_1 \in W$ and W is closed under scalar multiplication. Therefore, W is a subspace of $\mathbb{R}^3$. $\square$

Theorem 1.14

Theorem 1.14. Subspace Property of a Span
Let $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ be the span of $k > 0$ vectors in $\mathbb{R}^n$ Then W is a subspace of $\mathbb{R}^n$. (The vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are said to *span* or *generate* the subspace.)

Proof. We use Definition 1.16, "Closure and Subspace." Let $\vec{u}, \vec{v} \in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and let c be a scalar.

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$$\begin{aligned}
\vec{u} + \vec{v} &= (r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k) + (s_1\vec{w}_1 + s_2\vec{w}_2 + \cdots + s_k\vec{w}_k) \\
&= (r_1 + s_1)\vec{w}_1 + (r_2 + s_2)\vec{w}_2 + \cdots + (r_k + s_k)\vec{w}_k \text{ by S1 and S2} \\
&\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)
\end{aligned}$$

and so $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is closed under vector addition.

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$$\begin{aligned}
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Proof (continued). Next,

$$\begin{aligned} c\vec{u} &= c(r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k) \\ &= (cr_1)\vec{w}_1 + (cr_2)\vec{w}_2 + \cdots + (cr_k)\vec{w}_k \text{ by S1 and S3} \\ &\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \end{aligned}$$

and so $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is closed under scalar multiplication. So by Definition 1.16 $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is a subspace of $\mathbb{R}^n$. □

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Page 100 Number 18

Page 100 Number 18. Find a generating set for the solution set of the homogeneous linear system:

$$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & x_3 & - & x_4 & = & 0 \\ & & x_2 & + & x_3 & & & = & 0 \\ x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 0. \end{array}$$

Solution. We apply Gauss-Jordan elimination to the augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 3 & 0 \end{array}\right] \underset{R_3 \to R_3 - R_1}{\sim} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & 4 & 0 \end{array}\right]$$

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$$\underset{R_3 \to R_3 - 3R_2}{\underset{R_1 \to R_1 + R_2}{\sim}} \left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & 4 & 0 \end{array}\right] \underset{R_3 \to R_3/(-5)}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4/5 & 0 \end{array}\right]$$

...

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. . .

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Solution (continued). . . .

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4/5 & 0 \end{array}\right] \underset{R_2 \to R_2 - R_3}{\overset{R_1 \to R_1 - 2R_3}{\widetilde{}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3/5 & 0 \\ 0 & 1 & 0 & 4/5 & 0 \\ 0 & 0 & 1 & -4/5 & 0 \end{array}\right].$$

Returning to a system of equations,

$$\begin{aligned} x_1 + (3/5)x_4 &= 0 \\ x_2 + (4/5)x_4 &= 0 \\ x_3 - (4/5)x_4 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 &= -(3/5)x_4 \\ x_2 &= -(4/5)x_4 \\ x_3 &= (4/5)x_4 \\ x_4 &= x_4. \end{aligned}$$

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$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4/5 & 0 \end{array}\right] \underset{R_2 \to R_2 - R_3}{\overset{R_1 \to R_1 - 2R_3}{\widetilde{}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3/5 & 0 \\ 0 & 1 & 0 & 4/5 & 0 \\ 0 & 0 & 1 & -4/5 & 0 \end{array}\right].$$

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So let $r = x_4$ be a free variable and we have that the general solution is of the form $\vec{x} \in \left\{ r \begin{bmatrix} -3/5 \\ -4/5 \\ 4/5 \\ 1 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$.

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$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4/5 & 0 \end{array}\right] \underset{R_2 \to R_2 - R_3}{\overset{R_1 \to R_1 - 2R_3}{\widetilde{}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3/5 & 0 \\ 0 & 1 & 0 & 4/5 & 0 \\ 0 & 0 & 1 & -4/5 & 0 \end{array}\right].$$

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Solution (continued). So a generating set for the system is

$$\boxed{\left\{\begin{bmatrix} -3/5 \\ -4/5 \\ 4/5 \\ 1 \end{bmatrix}\right\}.}$$

Note: We could have let $s = x_4/5$ be a free variable in which case a generating set is given by the simpler $\left\{\begin{bmatrix} -3 \\ -4 \\ 4 \\ 5 \end{bmatrix}\right\}$. $\square$

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Theorem 1.15

Theorem 1.15. Unique Linear Combinations.
The set $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ if and only if

$$r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$$

implies

$$r_1 = r_2 = \cdots = r_k = 0.$$

Proof. Suppose $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$. Then by Definition 1.17, "Basis for a Subspace," every vector in W can be expressed uniquely as a linear combination of the $\vec{w}_i$.

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Proof (continued). Now suppose that $r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$ implies that $r_1 = r_2 = \cdots = r_k = 0$. Let $\vec{w} \in W$ and suppose $\vec{w} = c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_k\vec{w}_k = d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_k\vec{w}_k$. Then $\vec{0} = \vec{w} - \vec{w} = (c_1 - d_1)\vec{w}_1 + (c_2 - d_2)\vec{w}_2 + \cdots + (c_k - d_k)\vec{w}_k$ (by S1 and S2). By hypothesis for this case, we must have $c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0$.

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implies

$$r_1 = r_2 = \cdots = r_k = 0.$$

Proof (continued). Now suppose that $r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$ implies that $r_1 = r_2 = \cdots = r_k = 0$. Let $\vec{w} \in W$ and suppose $\vec{w} = c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_k\vec{w}_k = d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_k\vec{w}_k$. Then $\vec{0} = \vec{w} - \vec{w} = (c_1 - d_1)\vec{w}_1 + (c_2 - d_2)\vec{w}_2 + \cdots + (c_k - d_k)\vec{w}_k$ (by S1 and S2). By hypothesis for this case, we must have $c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0$. That is, we must have $c_1 = d_1$, $c_2 = d_2$, ..., $c_k = d_k$. Hence every vector of W is a unique linear combination of the $\vec{w}_i$, as claimed. □

Page 100 Number 22(a)

Page 100 Number 22(a). Use Theorem 1.15 to determine whether the set $\{[-1, 1], [1, 2]\}$ is a basis for the subspace of $\mathbb{R}^2$ that it spans.

Solution. Based on Theorem 1.15, we consider scalars $r_1, r_2 \in \mathbb{R}$ such that $r_1[-1, 1] + r_2[1, 2] = [0, 0]$. This implies $[-r_1, r_1] + [r_2, 2r_2] = [0, 0]$ or $[-r_1 + r_2, r_1 + 2r_2] = [0, 0]$.

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$$-r_1 + r_2 = 0 \qquad (1)$$
$$r_1 + 2r_2 = 0. \qquad (2)$$

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Page 100 Number 22(b)

Page 100 Number 22(b). Use Theorem 1.16 to determine whether the set $\{[-1,1],[1,2]\}$ is a basis for the subspace of $\mathbb{R}^2$ that it spans.

Solution. We define matrix A which has as its *columns* the vectors in the set: $A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$. By Theorem 1.16, we see that the columns of A form a basis for $\mathbb{R}^2$ if and only if A is row equivalent to $\mathcal{I}$.

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$$A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \underset{R_2 \to R_2 + R_1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

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$$\overset{R_2 \to R_2/3}{\widetilde{}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \overset{R_1 \to R_1 - 2R_2}{\widetilde{}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{I}.$$

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So $A \sim \mathcal{I}$ and hence the columns of A form a basis for $\mathbb{R}^2$; that is, the set $\{[-1,1],[1,2]\}$ is a basis for the subspace of $\mathbb{R}^2$ that it spans. (Since there are two vectors, their span is all of $\mathbb{R}^2$.) □

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Page 97 Example 6

Example. Page 97 Example 6. A basis of $\mathbb{R}^n$ cannot contain more than n vectors.

Proof. Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for $\mathbb{R}^n$ and ASSUME $k > n$. Consider the system $A\vec{x} = \vec{0}$ where the column vectors of A are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. Then A has n rows and k columns (corresponding to n equations in k unknowns).

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Theorem 1.18

Theorem 1.18. Structure of the Solution Set of $A\vec{x} = \vec{b}$.
Let $A\vec{x} = \vec{b}$ be a linear system. If $\vec{p}$ is any particular solution of $A\vec{x} = \vec{b}$ and $\vec{h}$ is a solution to $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$. In fact, every solution of $A\vec{x} = \vec{b}$ has the form $\vec{p} + \vec{h}$ and the general solution is $\vec{x} = \vec{p} + \vec{h}$ where $A\vec{h} = \vec{0}$ (that is, $\vec{h}$ is an arbitrary element of the nullspace of A).

Proof. Let $\vec{p}$ be a particular solution of $A\vec{x} = \vec{b}$ (so that $A\vec{p} = \vec{b}$). Let $\vec{h}$ be a solution of the homogeneous system $A\vec{x} = \vec{0}$ (so that $A\vec{h} = \vec{0}$).

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$$\begin{aligned} A(\vec{p} + \vec{h}) &= A\vec{p} + A\vec{h} \text{ by Theorem 1.2.A(10),} \\ &\quad \text{Distribution of Matrix Multiplication} \\ &= \vec{b} + \vec{0} = \vec{b}. \end{aligned}$$

So $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$.

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Proof (continued). Now suppose $\vec{q}$ is any solution to $A\vec{x} = \vec{b}$. With $\vec{p}$ as a particular solution to $A\vec{x} = \vec{b}$ we have

$$\begin{aligned} A(\vec{q} - \vec{p}) &= A\vec{q} - A\vec{p} \text{ by Theorem 1.2.A(10),} \\ &\quad \text{Distribution of Matrix Multiplication} \\ &= \vec{b} - \vec{b} = \vec{0}. \end{aligned}$$

Theorem 1.18 (continued)

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So $\vec{q} - \vec{p}$ is a solution of $A\vec{x} = \vec{0}$, say $\vec{q} - \vec{p} = \vec{h}$. So $\vec{q} = \vec{p} + \vec{h}$ and every solution $\vec{x}$ of $A\vec{x} = \vec{b}$ is of the form $\vec{p} + \vec{h}$ where $\vec{p}$ is a particular solution of $A\vec{x} + \vec{b}$ and $\vec{h}$ is any solution of $A\vec{x} = \vec{h}$. □

Theorem 1.18 (continued)

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Page 100 Number 36

Page 100 Number 36. Solve the linear system

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & 4 \\ 2x_1 & + & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ x_1 & - & 7x_2 & - & 6x_3 & + & 2x_4 & = & 6 \end{array}$$

and express the solution set in a form that illustrates Theorem 1.18.

Solution. We apply Gauss-Jordan elimination to the augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 4 \\ 2 & 1 & -3 & -1 & 6 \\ 1 & -7 & -6 & 2 & 6 \end{array}\right] \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 4 \\ 0 & 5 & -5 & -3 & -2 \\ 0 & -5 & -7 & 1 & 2 \end{array}\right] \overset{R_3 \to R_3 + R_2}{\sim}$$

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$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 4 \\ 0 & 5 & -5 & -3 & -2 \\ 0 & 0 & -12 & -2 & 0 \end{array}\right] \underset{R_3 \to R_3/(-12)}{\overset{R_2 \to R_2/5}{\sim}} \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 4 \\ 0 & 1 & -1 & -3/5 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right]$$

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Page 100 Number 36 (continued 1)

Solution (continued).

$$\overset{R_1 \to R_1 + 2R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1/5 & 16/5 \\ 0 & 1 & -1 & -3/5 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right]$$

$$\overset{\substack{R_1 \to R_1 + R_3 \\ R_2 \to R_2 + R_3}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/30 & 16/5 \\ 0 & 1 & 0 & -13/30 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right].$$

Page 100 Number 36 (continued 1)

Solution (continued).

$$\underset{R_1 \to R_1 + 2R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1/5 & 16/5 \\ 0 & 1 & -1 & -3/5 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right]$$

$$\underset{R_2 \to R_2 + R_3}{\underset{R_1 \to R_1 + R_3}{\sim}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/30 & 16/5 \\ 0 & 1 & 0 & -13/30 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right].$$

This corresponds to the system of equations:

$$\begin{array}{ccccccc} x_1 & & & - & (1/30)x_4 & = & 16/5 \\ & x_2 & & - & (13/30)x_4 & = & -2/5 \\ & & x_3 & + & (1/6)x_4 & = & 0 \end{array}$$

For a particular solution $\vec{p}$ to the original system of equations we choose to set $x_4 = 0$ so that. . .

Page 100 Number 36 (continued 1)

Solution (continued).

$$\underset{R_1 \to R_1 + 2R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1/5 & 16/5 \\ 0 & 1 & -1 & -3/5 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right]$$

$$\underset{\substack{R_1 \to R_1 + R_3 \\ R_2 \to R_2 + R_3}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/30 & 16/5 \\ 0 & 1 & 0 & -13/30 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{array}\right].$$

This corresponds to the system of equations:

$$\begin{array}{rcrcl} x_1 & & - & (1/30)x_4 & = & 16/5 \\ & x_2 & - & (13/30)x_4 & = & -2/5 \\ & & x_3 + & (1/6)x_4 & = & 0 \end{array}$$

For a particular solution $\vec{p}$ to the original system of equations we choose to set $x_4 = 0$ so that...

Page 100 Number 36 (continued 2)

Solution (continued). $\ldots \vec{p} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix}$. With A as the coefficient matrix, the homogeneous system $A\vec{x} = \vec{0}$ reduces to a similar system of equations but with only 0's on the right hand side:

$$\begin{array}{rcl} x_1 \quad\quad\quad - (1/30)x_4 & = & 0 \\ x_2 \quad - (13/30)x_4 & = & 0 \\ x_3 + (1/6)x_4 & = & 0 \end{array} \quad \text{or} \quad \begin{array}{l} x_1 = (1/30)x_4 \\ x_2 = (13/30)x_4 \\ x_3 = -(1/6)x_4 \\ x_4 = x_4 \end{array}$$

Page 100 Number 36 (continued 2)

Solution (continued). $\ldots \vec{p} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix}$. With A as the coefficient matrix, the homogeneous system $A\vec{x} = \vec{0}$ reduces to a similar system of equations but with only 0's on the right hand side:

$$\begin{array}{rrrrcl} x_1 & & & -\ (1/30)x_4 & = & 0 \\ & x_2 & & -\ (13/30)x_4 & = & 0 \\ & & x_3 & +\ (1/6)x_4 & = & 0 \end{array} \quad \text{or} \quad \begin{array}{l} x_1 = (1/30)x_4 \\ x_2 = (13/30)x_4 \\ x_3 = -(1/6)x_4 \\ x_4 = x_4 \end{array}$$

So x_4 acts as a free variable in the associated homogeneous system of equations. To simplify the numbers, we set $x_4 = 30r$ where $r \in \mathbb{R}$ (since r is any element of $\mathbb{R}$ then $30r$ is any element of $\mathbb{R}$, and conversely).

Page 100 Number 36 (continued 2)

Solution (continued). $\ldots \vec{p} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix}$. With A as the coefficient matrix, the homogeneous system $A\vec{x} = \vec{0}$ reduces to a similar system of equations but with only 0's on the right hand side:

$$\begin{array}{rrrrrcl} x_1 & & & - & (1/30)x_4 & = & 0 \\ & x_2 & & - & (13/30)x_4 & = & 0 \\ & & x_3 & + & (1/6)x_4 & = & 0 \end{array} \quad \text{or} \quad \begin{array}{l} x_1 = (1/30)x_4 \\ x_2 = (13/30)x_4 \\ x_3 = -(1/6)x_4 \\ x_4 = x_4 \end{array}$$

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Page 100 Number 36 (continued 3)

Solution (continued). This gives the solution to the homogeneous system as

$$\begin{aligned} x_1 &= (1/30)(30r) = r \\ x_2 &= (13/30)(30r) = 13r \\ x_3 &= -(1/6)(30r) = -5r \\ x_4 &= 30r. \end{aligned}$$

So the solution *set* to the homogeneous system of equations $A\vec{x} = \vec{b}$ is $\left\{ r \begin{bmatrix} 1 \\ 13 \\ -5 \\ 30 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$ (this is the nullspace of A).

Page 100 Number 36 (continued 3)

Solution (continued). This gives the solution to the homogeneous system as

$$\begin{aligned} x_1 &= (1/30)(30r) = r \\ x_2 &= (13/30)(30r) = 13r \\ x_3 &= -(1/6)(30r) = -5r \\ x_4 &= 30r. \end{aligned}$$

So the solution *set* to the homogeneous system of equations $A\vec{x} = \vec{b}$ is $\left\{ r \begin{bmatrix} 1 \\ 13 \\ -5 \\ 30 \end{bmatrix} \,\middle|\, r \in \mathbb{R} \right\}$ (this is the nullspace of A). Therefore, in the notation of Theorem 1.18, the general solutions to the original (nonhomogeneous) system of equations is

$$\vec{x} = \vec{p} + \vec{h} \text{ where } \vec{p} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{h} \in \left\{ r \begin{bmatrix} 1 \\ 13 \\ -5 \\ 30 \end{bmatrix} \,\middle|\, r \in \mathbb{R} \right\}. \quad \square$$

Page 100 Number 36 (continued 3)

Solution (continued). This gives the solution to the homogeneous system as

$$x_1 = (1/30)(30r) = r$$
$$x_2 = (13/30)(30r) = 13r$$
$$x_3 = -(1/6)(30r) = -5r$$
$$x_4 = 30r.$$

So the solution *set* to the homogeneous system of equations $A\vec{x} = \vec{b}$ is $\left\{ r \begin{bmatrix} 1 \\ 13 \\ -5 \\ 30 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$ (this is the nullspace of A). Therefore, in the notation of Theorem 1.18, the general solutions to the original (nonhomogeneous) system of equations is

$$\boxed{\vec{x} = \vec{p} + \vec{h} \text{ where } \vec{p} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{h} \in \left\{ r \begin{bmatrix} 1 \\ 13 \\ -5 \\ 30 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}.} \quad \square$$

Page 101 Number 43

Page 101 Number 43. Use Theorem 1.18 to prove why no system of linear equations can have exactly two solutions.

Proof. ASSUME to the contrary that linear system $A\vec{x} = \vec{b}$ does have exactly two solutions, say $\vec{p}_1$ and $\vec{p}_2$ (where $\vec{p}_1 \neq \vec{p}_2$).

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Page 101 Number 43

Page 101 Number 43. Use Theorem 1.18 to prove why no system of linear equations can have exactly two solutions.

Proof. ASSUME to the contrary that linear system $A\vec{x} = \vec{b}$ does have exactly two solutions, say $\vec{p}_1$ and $\vec{p}_2$ (where $\vec{p}_1 \neq \vec{p}_2$). Then $A(\vec{p}_1 - \vec{p}_2) = A\vec{p}_1 - A\vec{p}_2 = \vec{b} - \vec{b} = \vec{0}$ and so $\vec{h} = \vec{p}_1 - \vec{p}_2 \neq 0$ is a solution to the homogeneous system $A\vec{x} = \vec{0}$. Now $\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_1$ (since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_1$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction) and $\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_2$ (since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_2$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction). So by Theorem 1.18, $\vec{p}_1 + \vec{h}$ is a third solution to $A\vec{x} = \vec{b}$. This is a CONTRADICTION to the hypotheses. So the assumption that $A\vec{x} = \vec{b}$ has exactly two solutions is false and the claim follows. □

Page 101 Number 47

Page 101 Number 47. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^n$.

Proof. We use Definition 1.16, "Closure and Subspace." Let $\vec{u}, \vec{v} \in W_1 \cap W_2$.

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Proof. We use Definition 1.16, "Closure and Subspace." Let $\vec{u}, \vec{v} \in W_1 \cap W_2$. Since W_1 is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_1$. Since W_2 is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_2$.

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Now let r be a scalar. Since W_1 is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_1$. Since W_2 is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_2$.

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Now let r be a scalar. Since W_1 is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_1$. Since W_2 is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_2$. Hence $r\vec{u}$ is in both W_1 and W_2; that is, $r\vec{u} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under scalar multiplication. By Definition 1.16, $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$ □

Page 101 Number 47

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Proof. We use Definition 1.16, "Closure and Subspace." Let $\vec{u}, \vec{v} \in W_1 \cap W_2$. Since W_1 is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_1$. Since W_2 is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_2$. Hence $\vec{u} + \vec{v}$ is in both W_1 and W_2; that is, $\vec{u} + \vec{v} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under vector addition.

Now let r be a scalar. Since W_1 is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_1$. Since W_2 is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_2$. Hence $r\vec{u}$ is in both W_1 and W_2; that is, $r\vec{u} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under scalar multiplication. By Definition 1.16, $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$ □