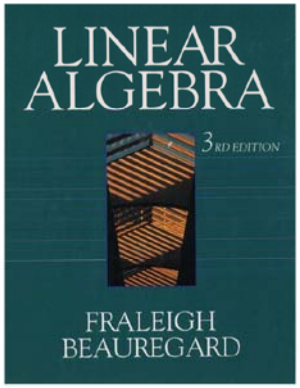
Linear Algebra

Chapter 1. Vectors, Matrices, and Linear Systems

Section 1.7. Application to Population Distribution—Proofs of Theorems



Page 113 Number 34

Page 113 Number 34. Explain why the transition matrix for the genetic model described in the notes above is

$$T = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}.$$

Solution. An individual in state 1 (GG) will always contribute a G gene and the hybrid state (Gg) will contribute a G gene $1/2$ of the time and a g gene $1/2$ of the time. So a GG individual produces a GG offspring with probability $(1)(1/2) = 1/2 = t_{11}$; a GG individual produces a Gg offspring with probability $(1)(1/2) = 1/2 = t_{21}$, and a GG individual cannot produce a gg offspring offspring and so $t_{31} = 0$.

Similarly, a gg individual cannot produce a GG offspring ($t_{13} = 0$), produces a Gg offspring with probability $(1/2)(1) = 1/2 = t_{23}$ and produces a gg offspring with probability $(1)(1/2) = 1/2 = t_{33}$.

Page 113 Number 34 (continued)

Page 113 Number 34. Explain why the transition matrix for the genetic model described in the notes above is

$$T = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}.$$

Solution (continued). Two Gg individuals produce a GG offspring when both contribute a G gene which happens with probability $(1/2)(1/2) = 1/4 = t_{12}$ and two Gg individuals produce a gg offspring when both contribute a g gene which happens with probability $(1/2)(1/2) = 1/4 = t_{32}$. A Gg individual can produce a Gg offspring in two ways: by contributing a G gene while the other parent contributes a g gene and vise versa. So $t_{22} = (1/2)(1/2) + (1/2)(1/2) = 1/2$. Hence, the transition matrix is as claimed. □

Page 113 Number 35

Page 113 Number 35. What proportion of the third-generation offspring (after two time periods) of the homozygous recessive gg population has produced homozygous dominant GG "grandchildren"?

Solution. We square the transition matrix to reflect two generations and consider the $(1, 3)$ entry:

$$T^2 = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}^2 = \begin{bmatrix} 3/8 & 1/4 & 1/8 \\ 1/2 & 1/2 & 1/2 \\ 1/8 & 1/4 & 3/8 \end{bmatrix},$$

so the answer is $\boxed{1/8.}$ □

Page 113 Number 36

Page 113 Number 36. What population of the third generation offspring (after two time periods) of the heterozygote population Gg has produced nonheterozygous "grandchildren"?

Solution. We consider T^2 given in Number 35 again,

$$T = \begin{bmatrix} 3/8 & 1/4 & 1/8 \\ 1/2 & 1/2 & 1/2 \\ 1/8 & 1/4 & 3/8 \end{bmatrix}.$$

We have the $(1,2)$ entry of T^2 as the proportion of the heterozygote's grandchildren which are GG, and the $(3,2)$ entry of T^2 as the proportion of the heterozygote's grandchildren which are gg. So the proportion of nonheterozygous grandchildren is $1/4 + 1/4 = \boxed{1/2.}$ □

Page 114 Number 37

Page 114 Number 37. If initially the entire population is hybrid, find the population distribution vector in the next generation.

Solution. We have initially $\vec{p} = [0,1,0]^T$ and in the next generation the population distribution vector is

$$T\vec{p} = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}}.$$

□

Page 114 Number 38

Page 114 Number 38. If initially the population is evenly divided among the three states, find the population distribution vector in the third generation (after two time periods)?

Solution. We have initially $\vec{p} = [1/3, 1/3, 1/3]^T$ and after two time periods the population distribution vector is

$$T^2\vec{p} = \begin{bmatrix} 3/8 & 1/4 & 1/8 \\ 1/2 & 1/2 & 1/2 \\ 1/8 & 1/4 & 3/8 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \boxed{\begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}}.$$

□

Page 112 Number 24

Page 112 Number 24. Find the steady-state distribution vector $\vec{s}$ of the regular transition matrix $T = \begin{bmatrix} 1/5 & 1/5 & 1/3 \\ 2/5 & 1/5 & 1/3 \\ 2/5 & 3/5 & 1/3 \end{bmatrix}$ for the resulting regular Markov chain.

Solution. Since no entry of T is 0 then T is a regular transition matrix. So, by Note 1.7.A, we consider the system of equations $(T - \mathcal{I})\vec{s} = \vec{0}$, which has the augmented matrix

$$\left[\begin{array}{ccc|c} -4/5 & 1/5 & 1/3 & 0 \\ 2/5 & -4/5 & 1/3 & 0 \\ 2/5 & 3/5 & -2/3 & 0 \end{array}\right] \underset{R_3 \to 15R_3}{\overset{R_1 \to 15R_1}{\underset{R_2 \to 15R_2}{\sim}}} \left[\begin{array}{ccc|c} -12 & 3 & 5 & 0 \\ 6 & -12 & 5 & 0 \\ 6 & 9 & -10 & 0 \end{array}\right]$$

$$\overset{R_1 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 6 & 9 & -10 & 0 \\ 6 & -12 & 5 & 0 \\ -12 & 3 & 5 & 0 \end{array}\right] \underset{R_3 \to R_3 + 2R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \left[\begin{array}{ccc|c} 6 & 9 & -10 & 0 \\ 0 & -21 & 15 & 0 \\ 0 & 21 & -15 & 0 \end{array}\right]$$

Page 112 Number 24 (continued 1)

Solution (continued).

$$\overset{R_3 \to R_3 + R_2}{\sim} \left[\begin{array}{ccc|c} 6 & 9 & -10 & 0 \\ 0 & -21 & 15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_2 \to R_2/(-21)}{\sim} \left[\begin{array}{ccc|c} 6 & 9 & -10 & 0 \\ 0 & 1 & -5/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{R_1 \to R_1 - 9R_2}{\sim} \left[\begin{array}{ccc|c} 6 & 0 & -25/7 & 0 \\ 0 & 1 & -5/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1 \to R_1/6}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -25/42 & 0 \\ 0 & 1 & -5/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we need

$$\begin{array}{rcrcl} x_1 & & -(25/42)x_3 & = & 0 \\ & x_2 & -(5/7)x_3 & = & 0 \\ & & 0 & = & 0 \end{array} \quad \text{or} \quad \begin{array}{rcl} x_1 & = & (25/42)x_3 \\ x_2 & = & (5/7)x_3 \\ x_3 & = & x_3 \end{array}$$

or, with $t = (1/42)x_3$ as a free variable (so that $x_3 = 42t$), $\begin{array}{rcl} x_1 & = & 25t \\ x_2 & = & 30t \\ x_3 & = & 42t \end{array}$.

Page 112 Number 24 (continued 2)

Page 112 Number 24. Find the steady-state distribution vector $\vec{s}$ of the regular transition matrix $T = \begin{bmatrix} 1/5 & 1/5 & 1/3 \\ 2/5 & 1/5 & 1/3 \\ 2/5 & 3/5 & 1/3 \end{bmatrix}$ for the resulting regular Markov chain.

Solution (continued). ... $\begin{array}{rcl} x_1 & = & 25t \\ x_2 & = & 30t \\ x_3 & = & 42t \end{array}$. Since t is any element of $\mathbb{R}$ but we need for a population distribution vector that $x_1 + x_2 + x_3 = 1$, we must choose $t = 1/(25 + 30 + 42) = 1/97$. Then the steady state distribution vector is $\boxed{\vec{s} = [25/97, 30/97, 42/97].}$ □

Page 114 Number 39

Page 114 Number 39. Find the steady-state distribution vector $\vec{s}$ for the genetic model of Exercises 35–38.

Solution. By Example 1.7.A, we see that this model is a regular chain. So, by Note 1.7.A, we consider the system of equations $(T - \mathcal{I})\vec{s} = \vec{0}$ which has the augmented matrix

$$\left[\begin{array}{ccc|c} -1/2 & 1/4 & 0 & 0 \\ 1/2 & -1/2 & 1/2 & 0 \\ 0 & 1/4 & -1/2 & 0 \end{array}\right] \overset{\substack{R_1 \to -4R_1 \\ R_2 \to 2R_2 \\ R_3 \to 4R_3}}{\sim} \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$\overset{R_1 \to R_1 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \overset{R_2 \to R_2 - R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$\overset{R_3 \to R_3 + R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_2 \to -R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Page 114 Number 39 (continued)

Solution (continued). So we need $\begin{array}{rcrcl} x_1 & & -x_3 & = & 0 \\ & x_2 & -2x_3 & = & 0 \\ & & 0 & = & 0 \end{array}$ or

$\begin{array}{rcl} x_1 & = & x_3 \\ x_2 & = & 2x_3 \\ x_3 & = & x_3 \end{array}$ or, with $t = x_3$ as a free variable, $\begin{array}{rcl} x_1 & = & t \\ x_2 & = & 2t \\ x_3 & = & t \end{array}$. Since t is any element of $\mathbb{R}$, but we need for a population distribution vector $x_1 + x_2 + x_3 = 1$, we must choose $t = 1/4$. Then the steady state distribution vector is $\boxed{\vec{s} = [1/4, 1/2, 1/4].}$ □

Note. If you are familiar with population genetics then you might recognize this as an illustration of *Hardy-Weinberg equilibrium*. We set gene frequencies at 1/2 for both G and g. So at equilibrium the proportion of GG individuals in the population is $(1/2)^2 = 1/4$, the proportion of Gg individuals in the population is $(1/2)(1/2) + (1/2)(1/2) = 1/2$, and the proportion of gg individuals in the population is $(1/2)^2 = 1/4$.