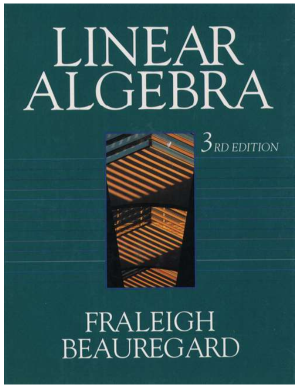
Linear Algebra

Chapter 2. Dimension, Rank, and Linear Transformations

Section 2.1. Independence and Dimension—Proofs of Theorems



Theorem 2.1

Theorem 2.1. Alternative Characterization of Basis
Let W be a subspace of $\mathbb{R}^n$. A subset $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ of W is a basis for W if and only if
(1) $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and
(2) the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent.

Proof. Recall that we defined $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ as a basis for W if every vector in W can be expressed as a unique linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ (see Definition 1.17).

Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ be a basis for W. Then every vector in W is a (unique) linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ and so these vectors span W and (1) holds. To show linear independence, we consider the equation $r_1\vec{w}_1 + r_2\vec{w}_2 r_k\vec{w}_k = \vec{0}$. We know that $r_1 = r_2 = \cdots = r_k = 0$ is one possible choice for the r_i, but since $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for W then this is the only choice for the r_i since $\vec{0}$ is a unique linear combination of the $\vec{w}_i$.

Theorem 2.1 (continued)

Proof (continued). That is, $r_1\vec{w}_1 + r_2\vec{w}_2 r_k\vec{w}_k = \vec{0}$ implies $r_1 = r_2 = \cdots = r_k = 0$. So, by Definition 2.1, "Linear Dependence and Independence," the $\vec{w}_i$ are not linearly dependent. That is, $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent and (2) holds.

Now suppose (1) and (2) hold. Then every vector in W can be expressed as some linear combination of the $\vec{w}_i$ since the $\vec{w}_i$ span W by (1). To show uniqueness of the linear combinations, suppose $\vec{v} \in W$ and $\vec{v} = r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = s_1\vec{w}_1 + s_2\vec{w}_2 + \cdots + s_k\vec{w}_k$. Then $(r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k) - (s_1\vec{w}_1 + s_2\vec{w}_2 + \cdots + s_k\vec{w}_k) = \vec{0}$ and $(r_1 - s_1)\vec{w}_1 + (r_2 - s_2)\vec{w}_2 + \cdots + (r_k - s_k)\vec{w}_k = \vec{0}$. Since the $\vec{w}_i$ are linearly independent by (2), then $r_1 - s_1 = r_2 - s_2 = \cdots = r_k - s_k = 0$ by Note 2.1.A and so $r_1 = s_1$, $r_2 = s_2$, $\ldots$, $r_k = s_k$. That is, there is a unique linear combination of the $\vec{w}_i$ which equals $\vec{v}$. Since $\vec{v}$ is an arbitrary vector in W, then $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for W. □

Page 134 Number 8

Page 134 Number 8. Use Theorem 2.1.A, "Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$," to find a basis for $W = \text{sp}([-3, 1], [9, -3])$.

Solution. We create matrix A with vectors $[-3, 1]$ and $[9, -3]$ as columns: $A = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix}$. Now we row reduce A:

$$A = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} \overset{R_1 \to R_1/(-3)}{\sim} \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \overset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} = H.$$

Since H is in row echelon form and has a pivot only in the first column, then by Theorem 2.1.A, $\boxed{\{[-3, 1]\} \text{ is a basis of } W.}$ □

Page 134 Number 10

Page 134 Number 10. Use Theorem 2.1.A, "Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$," to find a basis for $W = \text{sp}([-2,3,1],[3,-1,2],[1,2,3],[-1,5,4])$ in $\mathbb{R}^3$.

Solution. We create matrix A with the vectors in the spanning set as columns: $A = \begin{bmatrix} -2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Now we row reduce A:

$$A = \begin{bmatrix} -2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -1 & 2 & 5 \\ -2 & 3 & 1 & -1 \end{bmatrix}$$

$$\underset{R_3 \to R_3 + 2R_1}{\overset{R_2 \to R_2 - 3R_1}{\sim}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 7 & 7 & 7 \end{bmatrix} \underset{\sim}{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.$$

Page 134 Number 10 (continued)

Page 134 Number 10. Use Theorem 2.1.A, "Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$," to find a basis for $W = \text{sp}([-2,3,1],[3,-1,2],[1,2,3],[-1,5,4])$ in $\mathbb{R}^3$.

Solution (continued).

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.$$

Since H is in row echelon form and has a pivot in each of the first two columns then, by Theorem 2.1.A, a set consisting of the first two vectors, $\vec{w}_1, \vec{w}_2$ is a basis for W; that is, $\boxed{\{[-2,3,1],[3,-1,2]\} \text{ is a basis for } W.}$ Notice that the third vector is a linear combination of these two, $1[-2,3,1] + 1[3,-1,2] = [1,2,3]$, and the fourth vector is a linear combination of these two, $2[-2,3,1] + 1[3,-1,2] = [-1,5,4]$. $\square$

Page 135 Number 22

Page 135 Number 22. Determine whether the set $\{[1,-3,2],[2,-5,3],[4,0,1]\}$ is linearly dependent or independent.

Solution. We use Theorem 2.1.A, "Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$," and test to see if the set of vectors is a basis for its span. Let $W = \text{sp}([1,-3,2],[2,-5,3],[4,0,1])$. By Theorem 2.1, a basis for a vector space W is a linearly independent spanning set. Of course the set of vectors spans its span(!), so it is a basis of its span if and only if the set is a linearly independent set of vectors. We create matrix A with the vectors in the set as its columns and row reduce:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & -5 & 0 \\ 2 & 3 & 1 \end{bmatrix} \underset{R_3 \to R_3 - 2R_1}{\overset{R_2 \to R_2 + 3R_1}{\sim}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & -1 & -7 \end{bmatrix}$$

...

Page 135 Number 22 (continued)

Page 135 Number 22. Determine whether the set $\{[1,-3,2],[2,-5,3],[4,0,1]\}$ is linearly dependent or independent.

Solution (continued). ...

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & -1 & -7 \end{bmatrix} \underset{\sim}{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & 0 & 5 \end{bmatrix} = H.$$

Since H is in row echelon form and has a pivot in each column then by Theorem 2.1.A the set of all three vectors in $\{[1,-3,2],[2,-5,3],[4,0,1]\}$ form a basis for W. Therefore $\boxed{\text{the set of vectors is linearly independent.}}$ $\square$

Theorem 2.2

Theorem 2.2. Relative Sizes of Spanning and Independent Sets.
Let W be a subspace of $\mathbb{R}^n$. Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ be vectors in W that span W and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in W that are independent. Then $k \geq m$.

Proof. We give a proof by contradiction. ASSUME $k < m$. Since the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ span W and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are in W then there are scalars a_{ij} such that:

$$\begin{array}{rclcl} \vec{v}_1 & = & a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + & \cdots & +a_{k1}\vec{w}_k \\ \vec{v}_2 & = & a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + & \cdots & +a_{k2}\vec{w}_k \\ \vdots & & & \vdots & \\ \vec{v}_m & = & a_{1m}\vec{w}_1 + a_{2m}\vec{w}_2 + & \cdots & +a_{km}\vec{w}_k \end{array}.$$

Theorem 2.2 (continued 1)

Proof (continued). We introduce coefficients $x_1, x_2, \ldots, x_m$ of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ as follows:

$$\begin{array}{rclcl} x_1\vec{v}_1 & = & a_{11}x_1\vec{w}_1 + a_{21}x_1\vec{w}_2 + & \cdots & +a_{k1}x_1\vec{w}_k \\ x_2\vec{v}_2 & = & a_{12}x_2\vec{w}_1 + a_{22}x_2\vec{w}_2 + & \cdots & +a_{k2}x_2\vec{w}_k \\ \vdots & & & \vdots & \\ x_m\vec{v}_m & = & a_{1m}x_m\vec{w}_1 + a_{2m}x_m\vec{w}_2 + & \cdots & +a_{km}x_m\vec{w}_k \end{array}.$$

Now summing these equation we get

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m)\vec{w}_1$$

$$+(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m)\vec{w}_2 + \cdots + (a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{km}x_m)\vec{w}_k.$$

Consider the system of equations (which results by requiring each coefficient of the $\vec{w}_i$'s to be 0): ...

Theorem 2.2 (continued 2)

Proof (continued). ...

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m & = & 0 \\ & \vdots & \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{km}x_m & = & 0 \end{array}.$$

But this is then a system of k equations in m unknowns where $k < m$. By Corollary 2, "Fewer Equations than Unknowns, The Homogeneous Case," to Theorem 1.17, this system of equations has a nontrivial solution (that is, there are scalars $x_1, x_2, \ldots, x_m$ where some x_i is nonzero satisfying all m equations). But then we have $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{0}$ where some x_i is nonzero. This implies by Definition 2.1, "Linear Dependence and Independence," that the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly dependent, CONTRADICTING the hypothesis that the $\vec{v}_i$ are independent. So the assumption that $k < m$ is false and hence $k \geq m$, as claimed. □

Corollary 2.1.A

Corollary 2.1.A. Invariance of Dimension.
Any two bases of a subspace of $\mathbb{R}^n$ contains the same number of vectors.

Proof. Suppose that both B, a set of k vectors, and B', a set of m vectors, are bases for W. Then both B and B' are linearly independent spanning sets of W by Theorem 2.1, "Alternative Characterization of a Basis." This means that B is a set of k vectors spanning W and B' is a set of m linearly independent vectors in W. So by Theorem 2.2, "Relative Sizes of Spanning and Independent Sets," $k \geq m$. But also B' is a set of m vectors spanning W and B is a set of k linearly independent vectors in W. So by Theorem 2.2, $m \geq k$. Therefore $k = m$ and the bases B and B' have the same number of vectors. Since B and B' are arbitrary bases of W, the result follows. □

Theorem 2.3(1)

Theorem 2.3. Existence and Determination of Bases.
(1) Every subspace $W \neq \{\vec{0}\}$ of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.

Proof. Let W be a subspace of $\mathbb{R}^n$ where $W \neq \{\vec{0}\}$. Then there is some $\vec{w}_1 \in W$ such that $\vec{w}_1 \neq \vec{0}$. Set $B_1 = \{\vec{w}_1\}$. If $W = \text{sp}(\vec{w}_1)$ then stop, otherwise there is $\vec{w}_2 \in W$ such that $\vec{w}_2 \notin \text{sp}(\vec{w}_1)$. Set $B_2 = \{\vec{w}_1, \vec{w}_2\}$. Notice that $\vec{w}_1$ and $\vec{w}_2$ are linearly independent since $r_1\vec{w}_1 + r_2\vec{w}_2 = \vec{0}$ for $r_1 \neq 0$ implies $\vec{w}_1 = (-r_2/r_1)\vec{w}_2$, contradicting the choice of $\vec{w}_2 \notin \text{sp}(\vec{w}_1)$ (and similarly if $r_2 \neq 0$). If $W = \text{sp}(\vec{w}_1, \vec{w}_2)$ then stop. Otherwise, continue inductively so that if $W \neq \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$ where $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i$ are linearly independent, then there is $\vec{w}_{i+1} \in W$ such that $\vec{w}_{i+1} \notin \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$. Set $B_{i+1} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i, \vec{w}_{i+1}\}$. Then for

$$r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_i\vec{w}_i + r_{i+1}\vec{w}_{i+1} = \vec{0},$$

if $r_{i+1} \neq 0$ then $\vec{w}_{i+1} = (-r_1/r_{i+1})\vec{w}_1 + (-r_2/r_{i+1})\vec{w}_2 + \cdots + (-r_i/r_{i+1})\vec{w}_i$, contradicting the choice of $\vec{w}_{i+1} \notin \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$. So $r_{i+1} = 0$.

Theorem 2.3(1) (continued)

Theorem 2.3. Existence and Determination of Bases.
(1) Every subspace $W \neq \{\vec{0}\}$ of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.

Proof (continued). But then $r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_i\vec{w}_i = \vec{0}$ and so $r_1 = r_2 = \cdots = r_i = 0$ since $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i$ are linearly independent. So $r_1 = r_2 = \cdots = r_{i+1}$ and by Definition 2.1, "Linear Dependence and Independence," $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i, \vec{w}_{i+1}$ are linearly independent.

Now this process of creating linearly independent sets B_i consisting of i vectors must stop at some step $k \leq n$ by Theorem 2.2, "Relative Sizes of Spanning and Independent Sets." Then B_k is a linearly independent spanning set for W and so B_k is a basis for W by Theorem 2.1, "Alternative Characterization of Basis." □

Page 136 Number 34

Page 136 Number 34. Let $\vec{v}$ and $\vec{w}$ be independent column vectors in $\mathbb{R}^n$ and let A be an invertible $n \times n$ matrix where $n > 1$. Prove that the vectors $A\vec{v}$ and $A\vec{w}$ are independent.

Solution. We use Definition 2.1, "Linear Dependence and Independence," to test the set $\{A\vec{v}, A\vec{w}\}$ for linear independence. Suppose $r_1A\vec{v} + r_2A\vec{w} = \vec{0}$ for some $r_1, r_2 \in \mathbb{R}$. By Theorem 1.3.A, "Properties of Matrix Algebra," we have

$$\begin{aligned} \vec{0} &= r_1A\vec{v} + r_2A\vec{w} \\ &= A(r_1\vec{v}) + A(r_2\vec{w}) \text{ by Theorem 1.3.A(7)} \\ &= A(r_1\vec{v} + r_2\vec{w}) \text{ by Theorem 1.3.A(10).} \end{aligned}$$

Page 136 Number 34 (continued)

Page 136 Number 34. Let $\vec{v}$ and $\vec{w}$ be independent column vectors in $\mathbb{R}^n$ and let A be an invertible $n \times n$ matrix where $n > 1$. Prove that the vectors $A\vec{v}$ and $A\vec{w}$ are independent.

Solution (continued). Since A is invertible, then we can multiply both sides of this equation by A^{-1} to get

$$\begin{aligned} A^{-1}(\vec{0}) &= A^{-1}(A(r_1\vec{v} + r_2\vec{w})) \\ &= (A^{-1}A)(r_1\vec{v} + r_2\vec{w})) \text{ by Theorem 1.3.A(8)} \\ &= \mathcal{I}(r_1\vec{v} + r_2\vec{w}) = r_1\vec{v} + r_2\vec{w}. \end{aligned}$$

Therefore $\vec{0} = r_1\vec{v} + r_2\vec{w}$. Since $\vec{v}$ and $\vec{w}$ are independent then by Definition 2.1 we must have $r_1 = r_2 = 0$. That is, $r_1A\vec{v} + r_2A\vec{w} = \vec{0}$ implies $r_1 = r_2 = 0$. So, again by Definition 2.1, $\boxed{A\vec{v} \text{ and } A\vec{w} \text{ are independent.}}$ □

Page 136 Number 38

Page 136 Number 38. Prove that if W is a subspace of $\mathbb{R}^n$ and $\dim(W) = n$ then $W = \mathbb{R}^n$.

Solution. Of course $\dim(\mathbb{R}^n) = n$ since the standard basis for $\mathbb{R}^n$ (see Section 1.1) has n vectors. If W is a subspace of $\mathbb{R}^n$ of dimension n then by Definition 2.2, "Dimension of a Subspace," the number of elements in a basis B of W is n. By Theorem 2.1(2), "Alternative Characterization of a Basis," B is a linearly independent set. So B is a linearly independent set of n vectors and by Theorem 2.3(2), "Existence and Determination of Bases," B can be enlarged to become a basis for $\mathbb{R}^n$. However, a basis of $\mathbb{R}^n$ contains n vectors and so no additional vectors can be added to B in the creation of such a basis. So B must already be a basis of $\mathbb{R}^n$ and hence $W = \mathbb{R}^n$. $\square$