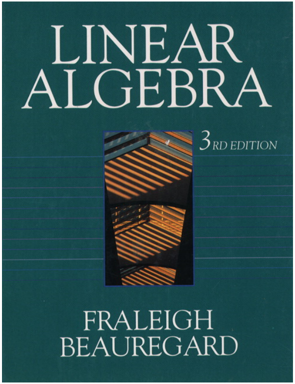
Linear Algebra

Chapter 2. Dimension, Rank, and Linear Transformations

Section 2.1. Independence and Dimension—Proofs of Theorems



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Theorem 2.1

Theorem 2.1. Alternative Characterization of Basis
Let W be a subspace of $\mathbb{R}^n$. A subset $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ of W is a basis for W if and only if
(1) $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and
(2) the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent.

Proof. Recall that we defined $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ as a basis for W if every vector in W can be expressed as a unique linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ (see Definition 1.17).

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Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ be a basis for W. Then every vector in W is a (unique) linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ and so these vectors span W and (1) holds.

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Proof (continued). That is, $r_1\vec{w}_1 + r_2\vec{w}_2 r_k\vec{w}_k = \vec{0}$ implies $r_1 = r_2 = \cdots = r_k = 0$. So, by Definition 2.1, "Linear Dependence and Independence," the $\vec{w}_i$ are not linearly dependent. That is, $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent and (2) holds.

Now suppose (1) and (2) hold. Then every vector in W can be expressed as some linear combination of the $\vec{w}_i$ since the $\vec{w}_i$ span W by (1).

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Now suppose (1) and (2) hold. Then every vector in W can be expressed as some linear combination of the $\vec{w}_i$ since the $\vec{w}_i$ span W by (1). To show uniqueness of the linear combinations, suppose $\vec{v} \in W$ and $\vec{v} = r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = s_1\vec{w}_1 + s_2\vec{w}_2 + \cdots + s_k\vec{w}_k$. Then $(r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k) - (s_1\vec{w}_1 + s_2\vec{w}_2 + \cdots + s_k\vec{w}_k) = \vec{0}$ and $(r_1 - s_1)\vec{w}_1 + (r_2 - s_2)\vec{w}_2 + \cdots + (r_k - s_k)\vec{w}_k = \vec{0}$.

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Solution. We create matrix A with vectors $[-3, 1]$ and $[9, -3]$ as columns:
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Solution. We create matrix A with the vectors in the spanning set as columns: $A = \begin{bmatrix} -2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

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$$\underset{\sim}{R_2 \to R_2 - 3R_1} \atop {R_3 \to R_3 + 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 7 & 7 & 7 \end{bmatrix} \underset{\sim}{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.$$

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Since H is in row echelon form and has a pivot in each of the first two columns then, by Theorem 2.1.A, a set consisting of the first two vectors, $\vec{w}_1, \vec{w}_2$ is a basis for W; that is, $\boxed{\{[-2,3,1],[3,-1,2]\} \text{ is a basis for } W.}$

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Page 135 Number 22. Determine whether the set $\{[1, -3, 2], [2, -5, 3], [4, 0, 1]\}$ is linearly dependent or independent.

Solution. We use Theorem 2.1.A, "Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$," and test to see if the set of vectors is a basis for its span. Let $W = \text{sp}([1, -3, 2], [2, -5, 3], [4, 0, 1])$.

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$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & -5 & 0 \\ 2 & 3 & 1 \end{bmatrix} \underset{R_3 \to R_3 - 2R_1}{\overset{R_2 \to R_2 + 3R_1}{\widetilde{}}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & -1 & -7 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & -1 & -7 \end{bmatrix} \underset{\sim}{\overset{R_3 \to R_3 + R_2}{}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & 0 & 5 \end{bmatrix} = H.$$

Since H is in row echelon form and has a pivot in each column then by Theorem 2.1.A the set of all three vectors in $\{[1, -3, 2], [2, -5, 3], [4, 0, 1]\}$ form a basis for W. Therefore $\boxed{\text{the set of vectors is linearly independent.}}$ □

Page 135 Number 22 (continued)

Page 135 Number 22. Determine whether the set $\{[1,-3,2],[2,-5,3],[4,0,1]\}$ is linearly dependent or independent.

Solution (continued). ...

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & -1 & -7 \end{bmatrix} \underset{\sim}{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & 0 & 5 \end{bmatrix} = H.$$

Since H is in row echelon form and has a pivot in each column then by Theorem 2.1.A the set of all three vectors in $\{[1,-3,2],[2,-5,3],[4,0,1]\}$ form a basis for W. Therefore $\boxed{\text{the set of vectors is linearly independent.}}$ $\square$

Theorem 2.2

Theorem 2.2. Relative Sizes of Spanning and Independent Sets. Let W be a subspace of $\mathbb{R}^n$. Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ be vectors in W that span W and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in W that are independent. Then $k \geq m$.

Proof. We give a proof by contradiction. ASSUME $k < m$.

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$$\begin{array}{rcll} \vec{v}_1 & = & a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + & \cdots \quad + a_{k1}\vec{w}_k \\ \vec{v}_2 & = & a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + & \cdots \quad + a_{k2}\vec{w}_k \\ \vdots & & & \vdots \\ \vec{v}_m & = & a_{1m}\vec{w}_1 + a_{2m}\vec{w}_2 + & \cdots \quad + a_{km}\vec{w}_k \end{array} .$$

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Theorem 2.2 (continued 1)

Proof (continued). We introduce coefficients $x_1, x_2, \ldots, x_m$ of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ as follows:

$$\begin{array}{ccccc} x_1\vec{v}_1 & = & a_{11}x_1\vec{w}_1 + a_{21}x_1\vec{w}_2 + & \cdots & +a_{k1}x_1\vec{w}_k \\ x_2\vec{v}_2 & = & a_{12}x_2\vec{w}_1 + a_{22}x_2\vec{w}_2 + & \cdots & +a_{k2}x_2\vec{w}_k \\ & \vdots & & \vdots & \\ x_m\vec{v}_m & = & a_{1m}x_m\vec{w}_1 + a_{2m}x_m\vec{w}_2 + & \cdots & +a_{km}x_m\vec{w}_k \end{array}.$$

Now summing these equation we get

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m)\vec{w}_1$$

$$+(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m)\vec{w}_2 + \cdots + (a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{km}x_m)\vec{w}_k.$$

Consider the system of equations (which results by requiring each coefficient of the $\vec{w}_i$'s to be 0): ...

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Theorem 2.2 (continued 2)

Proof (continued). ...

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m & = & 0 \\ & \vdots & \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{km}x_m & = & 0 \end{array}.$$

But this is then a system of k equations in m unknowns where $k < m$. By Corollary 2, "Fewer Equations than Unknowns, The Homogeneous Case," to Theorem 1.17, this system of equations has a nontrivial solution (that is, there are scalars $x_1, x_2, \ldots, x_m$ where some x_i is nonzero satisfying all m equations).

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Corollary 2.1.A

Corollary 2.1.A. Invariance of Dimension.
Any two bases of a subspace of $\mathbb{R}^n$ contains the same number of vectors.

Proof. Suppose that both B, a set of k vectors, and B', a set of m vectors, are bases for W. Then both B and B' are linearly independent spanning sets of W by Theorem 2.1, "Alternative Characterization of a Basis."

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Theorem 2.3(1)

Theorem 2.3. Existence and Determination of Bases.
(1) Every subspace $W \neq \{\vec{0}\}$ of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.

Proof. Let W be a subspace of $\mathbb{R}^n$ where $W \neq \{\vec{0}\}$. Then there is some $\vec{w}_1 \in W$ such that $\vec{w}_1 \neq \vec{0}$. Set $B_1 = \{\vec{w}_1\}$.

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Now this process of creating linearly independent sets B_i consisting of i vectors must stop at some step $k \leq n$ by Theorem 2.2, "Relative Sizes of Spanning and Independent Sets." Then B_k is a linearly independent spanning set for W and so B_k is a basis for W by Theorem 2.1, "Alternative Characterization of Basis." □

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Page 136 Number 34

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Solution. We use Definition 2.1, "Linear Dependence and Independence," to test the set $\{A\vec{v}, A\vec{w}\}$ for linear independence. Suppose $r_1 A\vec{v} + r_2 A\vec{w} = \vec{0}$ for some $r_1, r_2 \in \mathbb{R}$.

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$$\begin{aligned}\vec{0} &= r_1 A\vec{v} + r_2 A\vec{w} \\ &= A(r_1\vec{v}) + A(r_2\vec{w}) \text{ by Theorem 1.3.A(7)} \\ &= A(r_1\vec{v} + r_2\vec{w}) \text{ by Theorem 1.3.A(10).}\end{aligned}$$

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Solution (continued). Since A is invertible, then we can multiply both sides of this equation by A^{-1} to get

$$\begin{aligned} A^{-1}(\vec{0}) &= A^{-1}(A(r_1\vec{v} + r_2\vec{w})) \\ &= (A^{-1}A)(r_1\vec{v} + r_2\vec{w})) \text{ by Theorem 1.3.A(8)} \\ &= \mathcal{I}(r_1\vec{v} + r_2\vec{w}) = r_1\vec{v} + r_2\vec{w}. \end{aligned}$$

Therefore $\vec{0} = r_1\vec{v} + r_2\vec{w}$.

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Solution (continued). Since A is invertible, then we can multiply both sides of this equation by A^{-1} to get

$$\begin{aligned} A^{-1}(\vec{0}) &= A^{-1}(A(r_1\vec{v} + r_2\vec{w})) \\ &= (A^{-1}A)(r_1\vec{v} + r_2\vec{w})) \text{ by Theorem 1.3.A(8)} \\ &= \mathcal{I}(r_1\vec{v} + r_2\vec{w}) = r_1\vec{v} + r_2\vec{w}. \end{aligned}$$

Therefore $\vec{0} = r_1\vec{v} + r_2\vec{w}$. Since $\vec{v}$ and $\vec{w}$ are independent then by Definition 2.1 we must have $r_1 = r_2 = 0$. That is, $r_1 A\vec{v} + r_2 A\vec{w} = \vec{0}$ implies $r_1 = r_2 = 0$. So, again by Definition 2.1, $\boxed{A\vec{v} \text{ and } A\vec{w} \text{ are independent.}}$ □

Page 136 Number 34 (continued)

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Page 136 Number 38

Page 136 Number 38. Prove that if W is a subspace of $\mathbb{R}^n$ and $\dim(W) = n$ then $W = \mathbb{R}^n$.

Solution. Of course $\dim(\mathbb{R}^n) = n$ since the standard basis for $\mathbb{R}^n$ (see Section 1.1) has n vectors. If W is a subspace of $\mathbb{R}^n$ of dimension n then by Definition 2.2, "Dimension of a Subspace," the number of elements in a basis B of W is n.

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