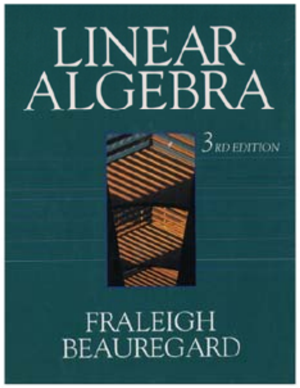
Linear Algebra

Chapter 2. Dimension, Rank, and Linear Transformations

Section 2.2. The Rank of a Matrix—Proofs of Theorems



Page 140 Number 6

Page 140 Number 6. Let $A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}$. Find (a) rank($A$), (b) a basis for the row space of A, (c) a basis for the column space of A, (d) a basis for the nullspace of A.

Solution. We apply the process of Note 2.2.A and row reduce A:

$$A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} -4 & 4 & 1 & 4 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R_1 \to R_1/(-4)}{\sim} \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \underset{\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 + 4R_1}}{\sim} \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 6 & 11/4 & 3 \\ 0 & -4 & 0 & -2 \end{bmatrix}$$

Page 140 Number 6 (continued 1)

Solution (continued).

$$\begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 6 & 11/4 & 3 \\ 0 & -4 & 0 & -2 \end{bmatrix} \underset{\substack{R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 + 2R_2}}{\sim} \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

$$\underset{R_4 \to R_4 + (24/25)R_3}{\sim} \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.$$

Since H is in row echelon form and has pivots in the first three columns we can apply Note 2.2.A to see that:

(a) $\boxed{\text{rank}(A) = 3}$ (the number of pivots in H),

Page 140 Number 6 (continued 2)

Solution (continued).
(b) a basis for the row space of A is the nonzero rows of

$$H = \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$\boxed{\{[1, -1, -1/4, -1], [0, 2, 3, 1], [0, 0, -25/4, 0]\}}$ (of course we could clean this up by multiplying the first and third vectors by 4 and getting the basis $\{[4, -4, -1, -4], [0, 2, 3, 1], [0, 0, -25, 0]\}$),

Page 140 Number 6 (continued 3)

Solution (continued).

(c) a basis for the column space of $A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}$ is given by the columns of A corresponding to columns of $H = \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which contain pivots, $\boxed{\left\{ \begin{bmatrix} 0 \\ -4 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}}$.

Page 140 Number 6 (continued 4)

Solution (continued). (d) For a basis for the nullspace of A, we consider the homogeneous system $A\vec{x} = \vec{0}$, which has (by Theorem 1.6) the same solution as $H\vec{x} = \vec{0}$. To simplify computations, we further row reduce the augmented matrix $[H \mid \vec{0}]$:

$$[H \mid \vec{0}] = \left[\begin{array}{cccc|c} 1 & -1 & -1/4 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & -25/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_3 \to (-4/25)R_3}{\overset{R_2 \to R_2/2}{\widetilde{\quad}}} \left[\begin{array}{cccc|c} 1 & -1 & -1/4 & -1 & 0 \\ 0 & 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1 \to R_1 + R_2}{\widetilde{\quad}} \cdots$$

Page 140 Number 6 (continued 5)

Solution (continued).

$$\cdots \overset{R_1 \to R_1 + R_2}{\widetilde{\quad}} \left[\begin{array}{cccc|c} 1 & 0 & 5/4 & -1/2 & 0 \\ 0 & 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_2 \to R_2 - (3/2)R_3}{\overset{R_1 \to R_1 - (5/4)R_3}{\widetilde{\quad}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Returning to a system of equations, ...

Page 140 Number 6 (continued 6)

Solution (continued). ...

$$\begin{array}{rrrrl} x_1 & & & -(1/2)x_4 & = 0 \\ & x_2 & & +(1/2)x_4 & = 0 \\ & & x_3 & & = 0 \\ & & & 0 & = 0 \end{array} \quad \text{or} \quad \begin{array}{l} x_1 = (1/2)x_4 \\ x_2 = -(1/2)x_4 \\ x_3 = 0 \\ x_4 = x_4. \end{array}$$

With $r = x_4/2$ as a free variable we have $\begin{array}{ll} x_1 & = (1/2)(2r) = r \\ x_2 & = (-1/2)(2r) = -r \\ x_3 & = 0 \\ x_4 & = 2r \end{array}$. So the general solution set for the system $A\vec{x} = \vec{0}$ is $\left\{ r \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$ and so a basis for the nullspace of A is $\boxed{\{[1, -1, 0, 2]^T\}}$. □

Page 141 Number 14

Page 141 Number 14. Let A and C be matrices such that the product AC is defined. Prove that the column space of AC is contained in the column space of A.

Solution. Let A be a $\ell \times m$ matrix, C a $m \times n$ matrix, and let $\vec{v} \in \mathbb{R}^\ell$ be in the column space of AC. Then $\vec{v}$ is a linear combination of the columns of AC by the definition of column space (see Section 1.6). So there is some vector $\vec{x} \in \mathbb{R}^m$ such that $(AC)\vec{x} = \vec{v}$, since $(AC)\vec{x}$ is a linear combination of the columns AC with coefficients as the components of $\vec{x}$ (see Note 1.3.A). Now $C\vec{x} \in \mathbb{R}^m$, say $\vec{y} = C\vec{x}$. But $\vec{v} = (AC)\vec{x} = A(C\vec{x}) = A\vec{y}$ and $A\vec{y}$ is a linear combination of the columns of A with coefficients as the components of $\vec{y}$. That is, $\vec{v}$ is in the column space of A. So any vector $\vec{v}$ in the column space of AC is in the column space of A, and the column space of A contains the column space of AC. □

Page 141 Number 18

Page 141 Number 18. Let A and C be matrices such that the product AC is defined. Prove that $\text{rank}(AC) \leq \text{rank}(A)$.

Solution. By the definition of rank, $\text{rank}(AC)$ is the dimension of the column space of AC and $\text{rank}(A)$ is the dimension of the column space of A. From Exercise 2.2.14 we see that the column space of AC is contained in the column space of A. That is, the column space of AC is a subspace of the column space of A. A basis of the column space of A consists of $\text{rank}(A)$ vectors and by Theorem 2.1(1), "Alternative Characterization of a Basis," these $\text{rank}(A)$ vectors span the column space of A. Now a basis of the column space of AC consists of $\text{rank}(AC)$ vectors and these $\text{rank}(AC)$ vectors are linearly independent by Theorem 2.1(2). So the basis of this column space of AC is a set of $\text{rank}(AC)$ linearly independent vectors in the column space of A and so by Theorem 2.2, "Relative Size of Spanning and Independent Sets," the size of a linearly independent set is less than or equal to the size of a spanning set; hence $\text{rank}(AC) \leq \text{rank}(A)$. □