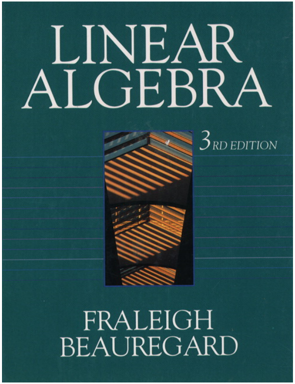
Linear Algebra

Chapter 2. Dimension, Rank, and Linear Transformations

Section 2.2. The Rank of a Matrix—Proofs of Theorems



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Page 140 Number 6. Let $A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}$. Find (a) rank($A$), (b) a basis for the row space of A, (c) a basis for the column space of A, (d) a basis for the nullspace of A.

Solution. We apply the process of Note 2.2.A and row reduce A:

$$A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \underset{\sim}{\overset{R_1 \leftrightarrow R_2}{}} \begin{bmatrix} -4 & 4 & 1 & 4 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}$$

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$$\underset{\sim}{R_1 \to R_1/(-4)} \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \underset{\sim}{R_3 \to R_3 - 3R_1} \; R_4 \to R_4 + 4R_1 \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 6 & 11/4 & 3 \\ 0 & -4 & 0 & -2 \end{bmatrix}$$

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Solution (continued).

$$\begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 6 & 11/4 & 3 \\ 0 & -4 & 0 & -2 \end{bmatrix} \underbrace{\overrightarrow{R_3 \to R_3 - 3R_2}}_{R_4 \to R_4 + 2R_2} \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

$$\underbrace{R_4 \to R_4 + (24/25) R_3} \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.$$

Since H is in row echelon form and has pivots in the first three columns we can apply Note 2.2.A to see that:

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Solution (continued).

$$\begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 6 & 11/4 & 3 \\ 0 & -4 & 0 & -2 \end{bmatrix} \underset{R_4 \to R_4 + 2R_2}{\overset{R_3 \to R_3 - 3R_2}{\sim}} \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

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Since H is in row echelon form and has pivots in the first three columns we can apply Note 2.2.A to see that:

(a) $\boxed{\text{rank}(A) = 3}$ (the number of pivots in H),

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Solution (continued).

$$\begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 6 & 11/4 & 3 \\ 0 & -4 & 0 & -2 \end{bmatrix} \underset{R_4 \to R_4 + 2R_2}{\overset{R_3 \to R_3 - 3R_2}{\widetilde{}}} \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

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Since H is in row echelon form and has pivots in the first three columns we can apply Note 2.2.A to see that:

(a) $\boxed{\text{rank}(A) = 3}$ (the number of pivots in H),

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Solution (continued).
(b) a basis for the row space of A is the nonzero rows of

$$H = \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$\boxed{\{[1,-1,-1/4,-1],[0,2,3,1],[0,0,-25/4,0]\}}$ (of course we could clean this up by multiplying the first and third vectors by 4 and getting the basis $\{[4,-4,-1,-4],[0,2,3,1],[0,0,-25,0]\}$),

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Solution (continued).

(c) a basis for the column space of $A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}$ is given by the columns of A corresponding to columns of $H = \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which contain pivots, $\boxed{\left\{ \begin{bmatrix} 0 \\ -4 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}.}$

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Solution (continued). (d) For a basis for the nullspace of A, we consider the homogeneous system $A\vec{x} = \vec{0}$, which has (by Theorem 1.6) the same solution as $H\vec{x} = \vec{0}$. To simplify computations, we further row reduce the augmented matrix $[H \mid \vec{0}]$:

$$[H \mid \vec{0}] = \left[\begin{array}{cccc|c} 1 & -1 & -1/4 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & -25/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_3 \to (-4/25)R_3}{\overset{R_2 \to R_2/2}{\sim}} \left[\begin{array}{cccc|c} 1 & -1 & -1/4 & -1 & 0 \\ 0 & 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1 \to R_1 + R_2}{\sim} \cdots$$

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$$[H \mid \vec{0}] = \left[\begin{array}{cccc|c} 1 & -1 & -1/4 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & -25/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underbrace{\overbrace{R_2 \to R_2/2}}_{R_3 \to (-4/25)R_3} \left[\begin{array}{cccc|c} 1 & -1 & -1/4 & -1 & 0 \\ 0 & 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overbrace{R_1 \to R_1 + R_2} \cdots$$

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Solution (continued).

$$\cdots \underset{R_1 \to R_1 + R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 5/4 & -1/2 & 0 \\ 0 & 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{\substack{R_1 \to R_1 - (5/4)R_3 \\ R_2 \to R_2 - (3/2)R_3}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Returning to a system of equations, . . .

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Solution (continued).

$$\cdots \underbrace{}_{R_1 \to R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 5/4 & -1/2 & 0 \\ 0 & 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underbrace{}_{\substack{R_1 \to R_1 - (5/4)R_3 \\ R_2 \to R_2 - (3/2)R_3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Returning to a system of equations, . . .

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Solution (continued). ...

$$\begin{array}{cccccc} x_1 & & & -(1/2)x_4 & = 0 \\ & x_2 & & +(1/2)x_4 & = 0 \\ & & x_3 & & = 0 \\ & & & 0 & = 0 \end{array} \quad \text{or} \quad \begin{array}{l} x_1 = (1/2)x_4 \\ x_2 = -(1/2)x_4 \\ x_3 = 0 \\ x_4 = x_4. \end{array}$$

With $r = x_4/2$ as a free variable we have $\begin{array}{ll} x_1 & = (1/2)(2r) = r \\ x_2 & = (-1/2)(2r) = -r \\ x_3 & = 0 \\ x_4 & = 2r \end{array}$. So the general solution set for the system $A\vec{x} = \vec{0}$ is $\left\{ r \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$ and so a basis for the nullspace of A is $\boxed{\{[1, -1, 0, 2]^T\}.}$ □

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Solution (continued). ...

$$\begin{array}{ccccl} x_1 & & & -(1/2)x_4 & = 0 \\ & x_2 & & +(1/2)x_4 & = 0 \\ & & x_3 & & = 0 \\ & & & 0 & = 0 \end{array} \quad \text{or} \quad \begin{array}{l} x_1 = (1/2)x_4 \\ x_2 = -(1/2)x_4 \\ x_3 = 0 \\ x_4 = x_4. \end{array}$$

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Page 141 Number 14

Page 141 Number 14. Let A and C be matrices such that the product AC is defined. Prove that the column space of AC is contained in the column space of A.

Solution. Let A be a $\ell \times m$ matrix, C a $m \times n$ matrix, and let $\vec{v} \in \mathbb{R}^\ell$ be in the column space of AC. Then $\vec{v}$ is a linear combination of the columns of AC by the definition of column space (see Section 1.6).

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Page 141 Number 18

Page 141 Number 18. Let A and C be matrices such that the product AC is defined. Prove that $\text{rank}(AC) \leq \text{rank}(A)$.

Solution. By the definition of rank, $\text{rank}(AC)$ is the dimension of the column space of AC and $\text{rank}(A)$ is the dimension of the column space of A. From Exercise 2.2.14 we see that the column space of AC is contained in the column space of A. That is, the column space of AC is a subspace of the column space of A.

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