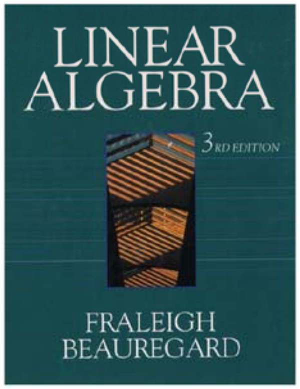
Linear Algebra

Chapter 2. Dimension, Rank, and Linear Transformations

Section 2.3. Linear Transformations of Euclidean Spaces—Proofs of Theorems



Page 153 Number 32

Page 153 Number 32. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that

$$T(r\vec{u} + s\vec{v}) = rT(\vec{u}) + sT(\vec{v})$$

for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and r and s. (As the text says, "linear transformations preserve linear combinations.")

Solution. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and let $r \in \mathbb{R}$ be a scalar. Then we have

$$\begin{aligned} T(r\vec{u} + s\vec{v}) &= T((r\vec{u}) + (s\vec{v})) = T(r\vec{u}) + T(s\vec{v}) \text{ by Definition 2.3(1), "Linear Transformation"} \\ &= rT(\vec{u}) + sT(\vec{v}) \text{ by Definition 2.3(2),} \end{aligned}$$

as claimed. □

Page 144 Example 3

Page 144 Example 3. Let A be an $m \times n$ matrix and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $T_A(\vec{x}) = A\vec{x}$ for each column vector $\vec{x} \in \mathbb{R}^n$. Prove that T_A is a linear transformation.

Solution. First, notice that for $m \times n$ matrix A and $n \times 1$ column vector in $\mathbb{R}^n$, we have that $A\vec{x}$ is in fact an $m \times 1$ column vector in $\mathbb{R}^m$. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and let $r \in \mathbb{R}$ be a scalar. Then we have

$$\begin{aligned} T_A(\vec{u} + \vec{v}) &= A(\vec{u} + \vec{v}) \text{ by the definition of } T_A \\ &= A\vec{u} + A\vec{v} \text{ by Theorem 1.3.A(10), "Distribution Laws"} \\ &= T_A(\vec{u}) + T_A(\vec{v}) \text{ by the definition of } T_A, \\ \text{and } T_A(r\vec{u}) &= A(r\vec{u}) \text{ by the definition of } T_A \\ &= rA\vec{u} \text{ by Theorem 1.3.A(7), "Scalars Pull Through"} \\ &= rT_A(\vec{u}) \text{ by the definition of } T_A. \end{aligned}$$

So T_A satisfies (1) and (2) of Definition 2.3, "Linear Transformation," and so T_A is a linear transformation. □

Page 152 Number 4

Page 152 Number 4. Is $T([x_1, x_2]) = [x_1 - x_2, x_2 + 1, 3x_1 - 2x_2]$ a linear transformation of $\mathbb{R}^2$ into $\mathbb{R}^3$? Why or why not?

Solution. We need to test T to see if it satisfies the definition of "linear transformation." Let $\vec{u} = [u_1, u_2], \vec{v} = [v_1, v_2] \in \mathbb{R}^2$. Then

$$T(\vec{u}) + T(\vec{v}) = T([u_1, u_2]) + T([v_1, v_2])$$

$$= [u_1 - u_2, u_2 + 1, 3u_1 - 2u_2] + [v_1 - v_2, v_2 + 1, 3v_1 - 2v_2)]$$

$$= [(u_1 - u_2) + (v_1 - v_2), (u_2 + 1) + (v_2 + 1), (3u_2 - 2u_2) + (3v_1 - 2v_2)]$$

and

$$T(\vec{u} + \vec{v}) = T([u_1, u_2] + [v_1, v_2]) = T([u_1 + v_1, u_2 + v_2])$$

$$= [(u_1 + v_1) - (u_2 + v_2), (u_2 + v_2) + 1, 3(u_1 + v_1) - 2(u_2 + v_2)]$$

$$= [(u_1 - u_2) + (v_1 - v_2), u_2 + v_2 + 1, (3u_1 - 2u_2) + (3v_1 - 2v_2)].$$

So $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ if and only if the components of these vectors are equal.

Page 152 Number 4 (continued)

Page 152 Number 4. If $T([x_1, x_2]) = [x_1 - x_2, x_2 + 1, 3x_1 - 2x_2]$ a linear transformation of $\mathbb{R}^2$ into $\mathbb{R}^3$? Why or why not?

Solution (continued). But the second component of $T(\vec{u}) + T(\vec{v}) = [(u_1 - u_2) + (v_1 - v_2), (u_2 + 1) + (v_2 + 1), (3u_2 - 2u_2) + (3v_1 - 2v_2)]$ is $u_2 + v_2 + 2$ and the second component of $T(\vec{u} + \vec{v}) = [(u_1 - u_2) + (v_1 - v_2), u_2 + v_2 + 1, (3u_1 - 2u_2) + (3v_1 - 2v_2)]$ is $u_2 + v_2 + 1$. So the second components are different and $T(\vec{u} + \vec{v}) \neq T(\vec{u}) + T(\vec{v})$, so T fails the definition of linear transformation and $\boxed{T \text{ is not a linear transformation.}}$ □

Page 145 Example 4

Page 145 Example 4. Determine all linear transformations of $\mathbb{R}$ into $\mathbb{R}$.

Solution. Let $T : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Denote $T([1])$ as $[m]$, that is, $T([1]) = [m]$. Then for any $[x] \in \mathbb{R}$ we have

$$\begin{aligned} T([x]) &= T([x1]) = xT([1]) \text{ by Definition 2.3(2)} \\ &= x[m] = [mx]. \end{aligned}$$

So if $T : \mathbb{R} \to \mathbb{R}$ is a linear transformation then $\boxed{T([x]) = [mx] \text{ for some } m \in \mathbb{R}.}$ □

Theorem 2.7

Theorem 2.7. Bases and Linear Transformations.
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $B = \{\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}\}$ be a basis for $\mathbb{R}^n$. For any vector $\vec{v} \in \mathbb{R}^n$, the vector $T(\vec{v})$ is uniquely determined by $T(\vec{b_1}), T(\vec{b_2}), \ldots, T(\vec{b_n})$..

Proof. Let $\vec{v} \in \mathbb{R}^n$. Since B is a basis, then by Definition 2.1, "Linear Dependence and Independence," there are unique scalars $r_1, r_2, \ldots, r_n \in \mathbb{R}$ such that $\vec{v} = r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_n\vec{b}_n$. Then by Exercise 32,

$$T(\vec{v}) = T(r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_n\vec{b}_n) = r_1T(\vec{b}_1) + r_2T(\vec{b}_2) + \cdots + r_nT(\vec{b}_n).$$

Since $r_1, r_2, \ldots, r_n$ are uniquely determined by $\vec{v}$, then $T(\vec{v})$ is completely determined by the vectors $T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)$. □

Corollary 2.3.A

Corollary 2.3.A. Standard Matrix Representation of Linear Transformations.
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear, and let A be the $m \times n$ matrix whose jth column is $T(\hat{e}_j)$. Then $T(\vec{x}) = A\vec{x}$ for each $\vec{x} \in \mathbb{R}^n$. A is the *standard matrix representation* of T.

Proof. Recall that with $\hat{e}_j$ as the jth standard basis vector of $\mathbb{R}^n$, we have $A\hat{e}_j$ is the jth column of A (see Note 1.3.A) and so $A\hat{e}_j = T(\hat{e}_j)$. If we define $T_A : \mathbb{R}^n \to \mathbb{R}^m$ as $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ then T_A is a linear transformation by Example 3 and T and T_A are the same on the standard basis $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ of $\mathbb{R}^n$. So by Theorem 2.7, "Bases and Linear Transformations," T and T_A are the same linear transformations mapping $\mathbb{R}^n \to \mathbb{R}^m$. That is, $T(\vec{x}) = T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$, as claimed. □

Page 152 Number 10

Page 152 Number 10. Assume that T is a linear transformation where $T([-1,1]) = [2,1,4]$ and $T([1,1]) = [-6,3,2]$. Find the standard matrix representation A_T of T and a (row) formula for $T([x,y])$.

Solution. We need to write the vector $[x,y]$ in terms of $[-1,1]$ and $[1,1]$. Notice that $-\frac{1}{2}[-1,1] + \frac{1}{2}[1,1] = [1,0]$ and $\frac{1}{2}[-1,1] + \frac{1}{2}[1,1] = [0,1]$. So by Corollary 2.3.A the columns of the standard matrix representation of T are $T([1,0])$ and $T([0,1])$. We have

$$T([1,0]) = T\left(-\frac{1}{2}[-1,1] + \frac{1}{2}[1,1]\right) = -\frac{1}{2}T([-1,1]) + \frac{1}{2}T([1,1])$$

$$= -\frac{1}{2}[2,1,4] + \frac{1}{2}[-6,3,2] = [-1,-1/2,-2] + [-3,3/2,1] = [-4,1,-1]$$

and $T([0,1]) = T\left(\frac{1}{2}[-1,1] + \frac{1}{2}[1,1]\right) = \frac{1}{2}T([-1,1]) + \frac{1}{2}T([1,1])$

$$= \frac{1}{2}[2,1,4] + \frac{1}{2}[-6,3,2] = [1,1/2,2] + [-3,3/2,1] = [-2,2,3].$$

Page 152 Number 10 (continued)

Page 152 Number 10. Assume that T is a linear transformation where $T([-1,1]) = [2,1,4]$ and $T([1,1]) = [-6,3,2]$. Find the standard matrix representation A_T of T and and a (row) formula for $T([x,y])$.

Solution (continued). So the matrix representation of T is

$$\boxed{A = \left[T([1,0])^T, T([0,1])^T\right] = \begin{bmatrix} -4 & -2 \\ 1 & 2 \\ -1 & 3 \end{bmatrix}}.$$ Therefore

$$T([x,y]) = A\vec{x} = \begin{bmatrix} -4 & -2 \\ 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4x-2y \\ x+2y \\ -x+3y \end{bmatrix}.$$

So $\boxed{T([x,y]) = [-4x-2y, x+2y, -x+3y].}$ □

Theorem 2.3.A

Theorem 2.3.A. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix representation A.
(1) The *range* $T[\mathbb{R}^n]$ of T is the column space of A.
(2) If W is a subspace of $\mathbb{R}^n$, then $T[W]$ is a subspace of $\mathbb{R}^m$ (i.e. T preserves subspaces).

Proof. (1) Recall that $T[\mathbb{R}^n] = \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$. Since A is the standard matrix representation of T then $T[\mathbb{R}^n] = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$. Now for $\vec{x} \in \mathbb{R}^n$, $A\vec{x}$ is a linear combination of the columns of A with the components of $\vec{x}$ as the coefficients (see Note 1.3.A) and conversely any linear combination of the columns of A equals $A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$ (namely, $\vec{x}$ with components equal to the coefficients in the linear combination). So the range of T, $T[\mathbb{R}^n]$, consists of precisely the same vectors as the column space of A.

Theorem 2.3.A (continued)

Theorem 2.3.A. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix representation A.
(1) The *range* $T[\mathbb{R}^n]$ of T is the column space of A.
(2) If W is a subspace of $\mathbb{R}^n$, then $T[W]$ is a subspace of $\mathbb{R}^m$ (i.e. T preserves subspaces).

Proof (continued). (2) Let W be a subspace of $\mathbb{R}^n$. Then W has a basis by Theorem 2.3(1), "Existence and Determination of Bases," say $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$. Now by Exercise 32,

$$T(r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_k\vec{b}_k) = r_1T(\vec{b}_1) + r_2T(\vec{b}_2) + \cdots + r_kT(\vec{b}_k)$$

for any $r_1, r_2, \ldots, r_k \in \mathbb{R}$. So $T[W] = \text{sp}(T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_k))$ and since the span of a set of vectors in $\mathbb{R}^m$ is a subspace of $\mathbb{R}^m$ by Theorem 1.14, "Subspace Property of a Span," we have that $T[W]$ is a subspace of $\mathbb{R}^m$. □

Theorem 2.3.B

Theorem 2.3.B. Matrix Multiplication and Composite Transformations.
A composition of two linear transformations T and T' with standard matrix representation A and A' yields a linear transformation $T' \circ T$ with standard matrix representation $A'A$.

Proof. We have that $T(\vec{x}) = A\vec{x}$ and $T'(\vec{y}) = A'\vec{y}$ for all appropriate $\vec{x}$ and $\vec{y}$ (that is, $\vec{x}$ is the domain of T and $\vec{y}$ in the domain of T'). Then for any $\vec{x}$ in the domain of T we have

$$\begin{aligned} (T' \circ T)(\vec{w}) &= T'(T(\vec{x})) \text{ by the definition of composition} \\ &= T'(A\vec{x}) \text{ since } T(\vec{x}) = A\vec{x} \\ &= A'(A\vec{x}) \text{ since } T'(\vec{y}) = A\prime\vec{y} \\ &= (A'A)\vec{x} \text{ by Theorem 1.3.A(8),} \\ &\quad \text{“Associativity of Matrix Multiplication”.} \end{aligned}$$

So the standard matrix representation of $T' \circ T$ is $A'A$, as claimed. □

Page 153 Number 20

Page 153 Number 20. If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined as $T([x_1, x_2]) = [2x_1 + x_2, x_1, x_1 - x_2]$ and $T' : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T'([x_1, x_2, x_3]) = [x_1 - x_2 + x_3, x_1 + x_2]$. Find the standard matrix representation for the linear transformation $T \circ T'$ that carries $\mathbb{R}^3$ into $\mathbb{R}^3$. Find a formula for $(T \circ T')([x_1, x_2, x_3])$.

Solution. First, we find the standard matrix representation of T and T'. We have $T([1, 0]) = [2, 1, 1]$ and $T([0, 1]) = [1, 0, -1]$, so by Corollary 2.3.A the standard matrix representation of T is $A_T = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$. Also, $T'([1, 0, 0]) = [1, 1]$, $T'([0, 1, 0]) = [-1, 1]$, and $T'([0, 0, 1]) = [1, 0]$ so the standard matrix representation of T' is $A_{T'} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Page 153 Number 20 (continued)

Solution (continued). By Theorem 2.3.B, the standard matrix representation of $T \circ T'$ is

$$A_T A_{T'} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & -1 & 2 \\ 1 & -1 & 1 \\ 0 & -2 & 1 \end{bmatrix}}.$$

By Corollary 2.3.A, a formula for $T \circ T'([x_1, x_2, x_3])$ can be found from

$$A_T A_{T'} \vec{x} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & -1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 + 2x_3 \\ x_1 - x_2 + x_3 \\ -2x_2 + x_3 \end{bmatrix},$$

and so $\boxed{T \circ T'([x_1, x_2, x_3]) = [3x_1 - x_2 + 2x_3, x_1 - x_2 + x_3, -2x_2 + x_3].}$ □

Page 153 Number 23

Page 153 Number 23. Consider the linear transformation $T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, x_1 + x_2, x_1]$. Find the standard matrix representation for T and determine if T is invertible. If it is, find a formula for T^{-1} in row notation.

Solution. We find the standard matrix representation for T using Corollary 2.3.A. We have $T([1, 0, 0]) = [1, 1, 1]$, $T([0, 1, 0]) = [1, 1, 0]$, and $T([0, 0, 1]) = [1, 0, 0]$. So the standard matrix representation for T is $\boxed{A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$. We test A for invertibility:

$$[A \mid \mathcal{I}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

Page 153 Number 23 (continued 1)

Solution (continued).

$$\underset{\sim}{R_2\leftrightarrow R_3}\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right] \underset{R_3\to -R_3}{\overset{R_2\to -R_2}{\sim}} \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\underset{\sim}{R_1\to R_1-R_2}\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \underset{\sim}{R_2\to R_2-R_3} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right].$$

So A is invertible and $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ and so $\boxed{T \text{ is invertible}}$ by Theorem 2.3.C.

Page 153 Number 23 (continued 2)

Page 153 Number 23. Consider the linear transformation $T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, x_1 + x_2, x_1]$. Find the standard matrix representation for T and determine if T is invertible. If it is, find a formula for T^{-1} in row notation.

Solution (continued). Also by Theorem 2.3.C, A^{-1} is the standard matrix representation of T^{-1} and we can find the formula for T^{-1} from

$$A^{-1}\vec{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 - x_3 \\ x_1 - x_2 \end{bmatrix}.$$

So $\boxed{T^{-1}([x_1, x_2, x_3]) = [x_3, x_2 - x_3, x_1 - x_2].}$ □