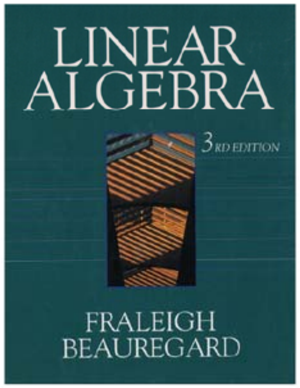
Linear Algebra

Chapter 2. Dimension, Rank, and Linear Transformations

Section 2.4. Linear Transformations of the Plane—Proofs of Theorems



Page 165 Number 4

Page 165 Number 4. Use the rotation matrix to derive trigonometric identities for $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Solution. Since $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ represents a rotation of $\mathbb{R}^2$ about the origin through an angle of θ, then A^3 represents a rotation of $\mathbb{R}^2$ about the origin through an angle 3θ. So

$$\begin{bmatrix} \cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^3$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^3\theta - \cos\theta\sin^2\theta - 2\cos\theta\sin^2\theta & -\cos^2\theta\sin\theta + \sin^3\theta - 2\cos^2\theta\sin\theta \\ 2\cos^2\theta\sin\theta + \cos^2\theta\sin\theta - \sin^3\theta & -2\cos\theta\sin^2\theta + \cos^3\theta - \cos\theta\sin^2\theta \end{bmatrix}.$$

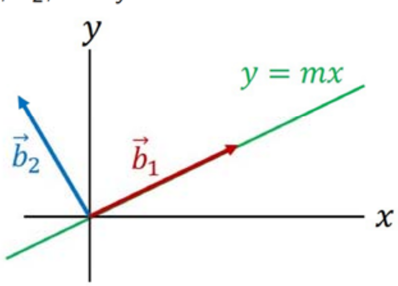
Hence $\boxed{\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \text{ and } \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.}$

□

Page 165 Number 6

Page 165 Number 6. Find the general matrix representation for the reflection of the plane about the line $y = mx$.

Solution. Let $\vec{b}_1 = [1, m]$ be a vector which, in standard position, lies along the line $y = mx$. Let $\vec{b}_2 = [-m, 1]$ so that $\vec{b}_2$ is orthogonal to $\vec{b}_1$. In standard position, $\vec{b}_1$, $\vec{b}_2$, and $y = mx$ are:



So $\vec{b}_1$ and $\vec{b}_2$ form a basis for $\mathbb{R}^2$ and by Theorem 2.7, "Bases and Linear Transformations," T is completely determined by $T(\vec{b}_1)$ and $T(\vec{b}_2)$.

Page 165 Number 6 (continued 1)

Solution (continued). Now we want matrix A where the first column of A is $T(\hat{e}_1) = T([1,0])$ and the second column of A is $T(\hat{e}_2) = T([0,1])$. Next, we need $\hat{e}_1$ and $\hat{e}_2$ in terms of $\vec{b}_1$ and $\vec{b}_2$. So we consider the system of equations $a_1\vec{b}_1 + a_2\vec{b}_2 = \hat{e}_1$ and $c_1\vec{b}_2 + c_2\vec{b}_2 = \hat{e}_2$. So we have $a_1[1, m] + a_2[-m, 1] = [a_1 - a_2 m, a_1 m + a_2] = [1, 0]$ and $c_1[1, m] + c_2[-m, 1] = [c_1 - c_2 m, c_1 m + c_2] = [0, 1]$, so we consider the augmented matrices:

$$\left[\begin{array}{cc|c} 1 & -m & 1 \\ m & 1 & 0 \end{array}\right] \underset{\sim}{\xrightarrow{R_2 \to R_2 - mR_1}} \left[\begin{array}{cc|c} 1 & -m & 1 \\ 0 & 1+m^2 & -m \end{array}\right] \underset{\sim}{\xrightarrow{R_2 \to R_2/(1+m^2)}}$$

$$\left[\begin{array}{cc|c} 1 & -m & 1 \\ 0 & 1 & -m/(1+m^2) \end{array}\right] \underset{\sim}{\xrightarrow{R_1 \to R_1 + mR_2}} \left[\begin{array}{cc|c} 1 & 0 & 1 - m^2/(1+m^2) \\ 0 & 1 & -m/(1+m^2) \end{array}\right],$$

so $a_1 = 1/(1+m^2)$ and $a_2 = -m/(1+m^2)$; and

Page 165 Number 6 (continued 2)

Solution (continued).

$$\left[\begin{array}{cc|c} 1 & -m & 0 \\ m & 1 & 1 \end{array}\right] \overset{R_2 \to R_2 - mR_1}{\sim} \left[\begin{array}{cc|c} 1 & -m & 0 \\ 0 & 1+m^2 & 1 \end{array}\right] \overset{R_2 \to R_2/(1+m^2)}{\sim}$$

$$\left[\begin{array}{cc|c} 1 & -m & 0 \\ 0 & 1 & 1/(1+m^2) \end{array}\right] \overset{R_1 \to R_1 + mR_2}{\sim} \left[\begin{array}{cc|c} 1 & 0 & m/(1+m^2) \\ 0 & 1 & 1/(1+m^2) \end{array}\right],$$

so $c_1 = m/(1+m^2)$ and $c_2 = 1/(1+m^2)$. Therefore,

$$T(\hat{e}_1) = T\left(\frac{1}{1+m^2}\vec{b}_1 - \frac{m}{1+m^2}\vec{b}_2\right) = \frac{1}{1+m^2}T(\vec{b}_1) - \frac{m}{1+m^2}T(\vec{b}_2)$$

$$= \frac{1}{1+m^2}\vec{b}_1 - \frac{m}{1+m^2}(-\vec{b}_2) = \frac{1}{1+m^2}[1, m] + \frac{m}{1+m^2}[-m, 1]$$

$$= \left[\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right], \ldots$$

Page 165 Number 6 (continued 3)

Solution (continued). . . . and

$$T(\hat{e}_2) = T\left(\frac{m}{1+m^2}\vec{b}_1 + \frac{1}{1+m^2}\vec{b}_2\right) = \frac{m}{1+m^2}T(\vec{b}_1) + \frac{1}{1+m^2}T(\vec{b}_2)$$

$$= \frac{m}{1+m^2}\vec{b}_1 + \frac{1}{1+m^2}(-\vec{b}_2) = \frac{m}{1+m^2}[1, m] - \frac{1}{1+m^2}[-m, 1]$$

$$= \left[\frac{2m}{1+m^2}, \frac{m^2-1}{1+m^2}\right].$$

So the matrix A representing T is

$$A = \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{bmatrix} = \boxed{\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}}.$$

□

Page 165 Number 8 (iii, iv)

Page 165 Number 8 (iii, iv). Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(iii) Show that T is a vertical expansion followed by a reflection about the x-axis if $r < -1$.

(iv) Show that T is a vertical contraction followed by a reflection about the x-axis if $-1 < r < 0$.

Solution. (iii) If $r < -1$ then $|r| > 1$ and so $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & |r| \end{bmatrix}$ is the standard matrix representation of a linear transformation T, which is a vertical expansion. Next, $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the standard matrix representation of a linear transformation T_1 which is a reflection about the x-axis. Now

$$XA_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & |r| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -|r| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$$

and so T is a vertical expansion followed by a reflection about the x-axis.

Page 165 Number 8 (iii, iv) (continued)

Page 165 Number 8 (iii, iv). Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(iv) Show that T is a vertical contraction followed by a reflection about the x-axis if $-1 < r < 0$.

Solution (continued). (iv) If $-1 < r < 0$ then $0 < |r| < 1$ and so $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & |r| \end{bmatrix}$ is the standard matrix representation of a linear transformation T_2, which is a vertical contraction. With X as in part (iii), we have

$$XA_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & |r| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -|r| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$$

and so T is a vertical contraction followed by a reflection about the x-axis. □

Theorem 2.4.A

Theorem 2.4.A. Geometric Description of Invertible Transformations of $\mathbb{R}^2$.
A linear transformation T of the plane $\mathbb{R}^2$ into itself is invertible if and only if T consists of a finite sequence of:
- Reflections in the x-axis, the y-axis, or the line $y = x$;
- Vertical or horizontal expansions or contractions; and
- Vertical or horizontal shears.

Proof. The three elementary row operations correspond to 2×2 matrices as follows:

(1) Row Interchange: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

(2) Row Scaling: $B_1 = \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$,

(3) Row Addition: $C_1 = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$.

Theorem 2.4.A (continued 1)

Proof (continued). Now A corresponds to reflection about the line $y = x$, B_1 with $r = -1$ corresponds to reflection about the y-axis, B_1 corresponds to a horizontal expansion if $r > 1$, B_1 corresponds to a horizontal contraction if $0 < r < 1$, B_1 corresponds to a horizontal expansion followed by a reflection about the y-axis if $r < -1$ (similar to Exercise 8(iii)), B_1 corresponds to a horizontal contraction followed by a reflection about the y-axis if $-1 < r < 0$ (similar to Exercise 8(iv)), B_2 with $r = -1$ corresponds to reflection about the x-axis, B_2 corresponds to a vertical expansion if $r > 1$, B_2 corresponds to a vertical contraction if $0 < r < 1$, B_2 corresponds to a vertical expansion followed by a reflection about the x-axis if $r < -1$ (as shown in Exercise 8(iii)), B_2 corresponds to a vertical contraction followed by a reflection about the x-axis if $-1 < r < 0$ (as shown in Exercise 8(iv)), C_1 corresponds to a vertical shear, and C_2 corresponds to a horizontal shear.

Theorem 2.4.A (continued 2)

Theorem 2.4.A. Geometric Description of Invertible Transformations of $\mathbb{R}^2$.
A linear transformation T of the plane $\mathbb{R}^2$ into itself is invertible if and only if T consists of a finite sequence of:
- Reflections in the x-axis, the y-axis, or the line $y = x$;
- Vertical or horizontal expansions or contractions; and
- Vertical or horizontal shears.

Proof (continued). Notice that this is an exhaustive list of all 2×2 elementary matrices and of reflections, expansions, contractions, and shears as listed in the statement of the theorem. The claim now follows. □

Page 165 Number 14

Page 165 Number 14. Consider $T([x, y]) = [x + y, 2x - y]$. Find the standard matrix representation and write it as a product of elementary matrices. Then describe T as a sequence of reflections, expansions, contractions, and shears.

Solution. First, $T([1, 0]) = [1, 2]$ and $T([0, 1]) = [1, -1]$, so the standard matrix representation of T is $\boxed{A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}}$. We use the technique of Section 1.5 to write A as a product of elementary matrices. We have

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \underset{\sim}{\overset{R_2 \to R_2 - 2R_1}{}} \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{\sim}{\overset{R_2 \to R_2 + 2R_1}{}} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = E_1^{-1},$$

$$\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \underset{\sim}{\overset{R_2 \to R_2/(-3)}{}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{\sim}{\overset{R_2 \to -3R_2}{}} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} = E_2^{-1},$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underset{\sim}{\overset{R_1 \to R_1 - R_2}{}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{\sim}{\overset{R_1 \to R_1 + R_2}{}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = E_3^{-1},$$

Page 165 Number 14 (continued)

Page 165 Number 14. Consider $T([x,y]) = [x+y, 2x-y]$. Find the standard matrix representation and write it as a product of elementary matrices. Then describe T as a sequence of reflections, expansions, contractions, and shears.

Solution (continued). So

$$\boxed{A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.}$$

So T consist of in order (reading from right to left) a horizontal shear, a vertical expansion and a reflection about the x-axis (see Exercise 8), and a vertical shear. □

Page 166 Number 18

Page 166 Number 18. Use algebraic properties of the dot product to compute $\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$, and prove from the resulting equation that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves length also preserves the dot product.

Solution. Let $\vec{u}$ and $\vec{v}$ be any vectors in $\mathbb{R}^2$. Then

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$
$$= \vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} = \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2.$$

Solving for $\vec{u}\cdot\vec{v}$ gives

$$\vec{u}\cdot\vec{v} = \frac{-1}{2}(\|\vec{u}-\vec{v}\|^2 - \|\vec{u}\|^2 - \|\vec{v}\|^2)$$
$$= \frac{1}{2}(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2).$$

Similarly, $T(\vec{u})\cdot T(\vec{v}) = \frac{1}{2}(\|T(\vec{u})\|^2 + \|T(\vec{v})\|^2 - \|T(\vec{u}) - T(\vec{v})\|^2)$.

Page 166 Number 18 (continued)

Page 166 Number 18. Use algebraic properties of the dot product to compute $\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdots(\vec{u}-\vec{v})$, and prove from the resulting equation that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves length also preserves the dot product.

Solution (continued). Now if T preserves lengths then $\|\vec{u}\| = \|T(\vec{u})\|$, $\|\vec{v}\| = \|T(\vec{v})\|$, and $\|T(\vec{u}-\vec{v})\| = \|\vec{u}-\vec{v}\|$. Hence

$$\begin{aligned} T(\vec{u})\cdot T(\vec{v}) &= \frac{1}{2}(\|T(\vec{u})\|^2 + \|T(\vec{v})\|^2 - \|T(\vec{u}) - T(\vec{v})\|^2) \\ &= \frac{1}{2}(\|T(\vec{u})\|^2 + \|T(\vec{v})\|^2 - \|T(\vec{u}-\vec{v})\|) \text{ since } T \text{ is linear} \\ &= \frac{1}{2}(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2) = \vec{u}\cdot\vec{v}. \end{aligned}$$

So T preserves dot products as claimed. □

Page 166 Number 20

Page 166 Number 20. Suppose that $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ preserves both length and angle. Prove that the two column vectors of the matrix A are orthogonal unit vectors.

Proof. Since A is the standard matrix representation of T, the columns of A are $T(\hat{e}_1) = T([1,0])$ and $T(\hat{e}_2) = T([0,1])$ by Corollary 2.3.A, "Standard Matrix Representation of Linear Transformations." Since T_A preserves lengths then $\|T(\hat{e}_1)\| = \|\hat{e}_1\| = 1$ and $\|T(\hat{e}_2)\| = \|\hat{e}_2\| = 1$, so the columns of A are unit vectors. Since T preserves angles and $\hat{e}_1 \perp \hat{e}_2$ then $T(\hat{e}_1) \perp T(\hat{e}_2)$; that is, the columns of A are orthogonal, as claimed. □