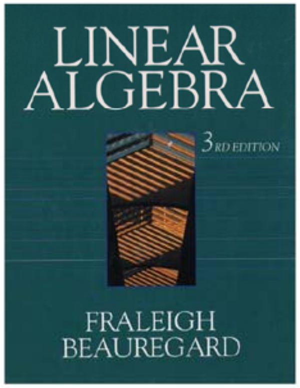
Linear Algebra

Chapter 2. Dimension, Rank, and Linear Transformations
Section 2.5. Lines, Planes, and Other Flats—Proofs of Theorems



Page 176 Number 8

Page 176 Number 8. Give parametric equations for the line in $\mathbb{R}^3$ through the point $(-1,3,0)$ with direction vector $\vec{d} = [-2,-1,4]$.

Solution. We introduce the translation vector $\vec{a}$ from $(0,0,0)$ to $(-1,3,0)$, so $\vec{a} = [-1-0, 3-0, 0-0] = [-1,3,0]$. Then the line is given parametrically as $\vec{x} = t\vec{d} + \vec{a}$ or

$$\boxed{\begin{array}{rclclcrcr} x_1 & = & td_1 + a_1 & = & a_1 + d_1 t & = & -1 & - & 2t \\ x_2 & = & td_2 + a_2 & = & a_2 + d_2 t & = & 3 & - & t \\ x_3 & = & td_3 + a_3 & = & a_3 + d_3 t & = & & & 4t \end{array}.}$$

□

Page 176 Number 12(a)

Page 176 Number 12(a). Consider the lines in $\mathbb{R}^3$ given parametrically by

$$\begin{array}{lcl} x_1 = 4 + t & \text{and} & x_1 = 11 + 3s \\ x_2 = 2 - 3t & & x_2 = -9 - 4s \\ x_3 = -3 + 5t & & x_3 = -4 - 3s \end{array}.$$

Determine whether the lines intersect. If they do intersect, find the point of intersection and determine whether the lines are orthogonal.

Solution. The lines intersect if there are t and s such that the first, second, and third coordinates of the points are the same. So we consider the system of equations

$$\begin{array}{rclcrcr} 4 + t & = & 11 + 3s & \text{or} & t - 3s & = & 7 \\ 2 - 3t & = & -9 - 4s & & -3t + 4s & = & -11 \\ -3 + 5t & = & -4 - 3s & & 5t + 3s & = & -1 \end{array}.$$

Page 176 Number 12(a) (continued)

Solution (continued). The associated augmented matrix for the system is

$$\left[\begin{array}{rr|r} 1 & -3 & 7 \\ -3 & 4 & -11 \\ 5 & 3 & -1 \end{array}\right] \underset{R_3 \to R_3 - 5R_1}{\overset{R_2 \to R_2 + 3R_1}{\widetilde{}}} \left[\begin{array}{rr|r} 1 & -3 & 7 \\ 0 & -5 & 10 \\ 0 & 18 & -36 \end{array}\right]$$

$$\underset{R_3 \to R_3/18}{\overset{R_2 \to R_2/5}{\widetilde{}}} \left[\begin{array}{rr|r} 1 & -3 & 7 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \end{array}\right] \underset{R_3 \to R_3 + R_2}{\overset{R_1 \to R_1 - 3R_2}{\widetilde{}}} \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right],$$

So we take $t = 1$ and $s = -2$ and
$\boxed{\text{the lines intersect at the point } (4+t, 2-3t, -3+5t)|_{t=1} = (5,-1,2).}$
The angle between the lines is the angle between their direction vectors. The components of the direction vectors are the coefficients of the parameters in the parametric equations, so the direction vectors are $[1,-3,5]$ and $[3,-4,-3]$. Since
$[1,-3,5] \cdot [3,-4,-3] = (1)(3) + (-3)(-4) + (5)(-3) = 0$,
$\boxed{\text{the lines are orthogonal.}}$ □

Page 177 Number 22

Page 177 Number 22. Find parametric equations of the plane that passes through the unit coordinate points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

Solution. We treat this as a 2-flat in $\mathbb{R}^3$. We can use the three non-colinear points to determine two vectors. The vector from point $(1,0,0)$ to $(0,1,0)$ is $[0-1, 1-0, 0-0] = [-1,1,0] = \vec{d}_1$ and the vector from point $(1,0,0)$ to $(0,0,1)$ is $[0-1, 0-0, 1-0] = [-1,0,1] = \vec{d}_2$. So we let $W = \text{sp}([-1,1,0],[-1,0,1])$. Since the plane passes through the point $(1,0,0)$ we can take $\vec{a} = [1,0,0]$. So as a k-flat, the plane is $\vec{a} + W = [1,0,0] + \text{sp}([-1,1,0],[-1,0,1])$. Parametrically, for (x_1,x_2,x_3) a point in the plane, the vector $\vec{x} = [x_1,x_2,x_3] \in \vec{a} + W$ and with t_1 and t_2 as parameters,
$[x_1,x_2,x_3] \in [1,0,0] + t_1[-1,1,0] + t_2[-1,0,1] = [1-t_1-t_2, t_1, t_2]$. So the parametric equations for the plane are

$$\boxed{\begin{array}{rcl} x_1 &=& 1 - t_1 - t_2 \\ x_2 &=& t_1 \\ x_3 &=& t_2 \end{array}}.$$

□

Page 177 Number 30

Page 177 Number 30. Find a (column) vector equation of the hyperplane that passes through the points $(1,2,1,2,3)$, $(0,1,2,1,3)$, $(0,0,3,1,2)$, $(0,0,0,1,4)$, and $(0,0,0,0,2)$ in $\mathbb{R}^5$.

Solution. For a hyperplane in $\mathbb{R}^5$ we need 4 basis vectors. As in Number 22, we take vectors between points for the basis vectors:

Tail of Vector	Head of Vector	Basis Vector
$(1,2,1,2,3)$	$(0,1,2,1,3)$	$\vec{d}_1 = [-1,-1,1,-1,0]^T$
$(1,2,1,2,3)$	$(0,0,3,1,2)$	$\vec{d}_2 = [-1,-2,2,-1,-1]^T$
$(1,2,1,2,3)$	$(0,0,0,1,4)$	$\vec{d}_3 = [-1,-2,-1,-1,1]^T$
$(1,2,1,2,3)$	$(0,0,0,0,2)$	$\vec{d}_4 = [-1,-2,-1,-2,-1]^T$

(We need the transposes to get column vectors, as directed.)

Page 177 Number 30 (continued 1)

Solution (continued). We define

$$W = \text{sp}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \\ -2 \\ -1 \end{bmatrix}\right).$$

We take as the translation vector $\vec{a}$ a vector from $(0,0,0,0,0)$ to one of the given points, say $(1,2,1,2,3)$, so that $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

Page 177 Number 30 (continued 2)

Solution. The hyperplane as a k-flat is then

$$\vec{a} + W = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} + \text{sp}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \\ -2 \\ -1 \end{bmatrix}\right).$$

As a vector equation, we introduce parameters t_1, t_2, t_3, and t_4 to get the vector equations:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} + t_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ -2 \\ 2 \\ -1 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ -2 \\ -1 \\ -1 \\ 1 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ -2 \\ -1 \\ -2 \\ -1 \end{bmatrix}}.$$

□

Page 177 Number 36

Page 177 Number 36. Solve the system of equations and express the solution set as a k-flat for:

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & 2x_3 & = & 4 \\ 2x_1 & + & 7x_2 & - & x_3 & = & -2 \\ x_1 & + & 3x_2 & + & x_3 & = & -6 \end{array}.$$

Solution. We consider the augmented matrix and reduce it:

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 4 \\ 2 & 7 & -1 & -2 \\ 1 & 3 & 1 & -6 \end{array}\right] \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - 2R_1}{\widetilde{}}} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 4 \\ 0 & -1 & 3 & -10 \\ 0 & -1 & 3 & -10 \end{array}\right]$$

$$\underset{R_3 \to R_3 - R_2}{\overset{R_1 \to R_1 + 4R_2}{\widetilde{}}} \left[\begin{array}{ccc|c} 1 & 0 & 10 & -36 \\ 0 & -1 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_2 \to -R_2}{\widetilde{}} \left[\begin{array}{ccc|c} 1 & 0 & 10 & -36 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so we need…

Page 177 Number 36 (continued)

Solution (continued). … $\begin{array}{rcrcr} x_1 & = & -36 & - & 10x_3 \\ x_2 & = & 10 & + & 3x_3 \\ x_3 & = & & & x_3 \end{array}$. With $t = x_3$ as a free variable, the general solution in vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -36 \\ 10 \\ 0 \end{bmatrix} + t \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}.$$ So we take $\vec{a} = \begin{bmatrix} -36 \\ 10 \\ 0 \end{bmatrix}$, $\vec{d}_1 = \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}$, and $W = \text{sp}\left(\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}\right)$. Then the k-flat is

$$\boxed{\vec{a} + W = \begin{bmatrix} -36 \\ 10 \\ 0 \end{bmatrix} + \text{sp}\left(\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}\right).}$$

□