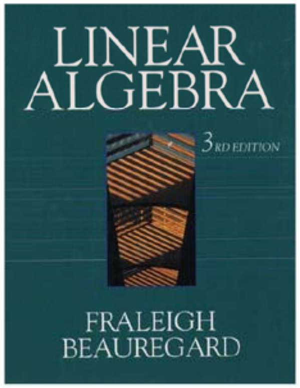
Linear Algebra

Chapter 3. Vector Spaces

Section 3.2. Basic Concepts of Vector Spaces—Proofs of Theorems



Theorem 3.2

Theorem 3.2. Test for Subspace.
A subset W of vector space V is a subspace if and only if
(1) $\vec{v}, \vec{w} \in W \Rightarrow \vec{v} + \vec{w} \in W$,
(2) for all $r \in \mathbb{R}$ and for all $\vec{v} \in W$, we have $r\vec{v} \in W$.

Proof. Let W be a subspace of V. W must be nonempty since $\vec{0}$ must be in W by Definition 3.1, "Vector Space." Also by Definition 3.1, we see that W must have a rule for adding two vectors $\vec{v}$ and $\vec{w}$ in W to produce a vector $\vec{v} + \vec{w}$. Addition in W is the same as in V, so it is necessary that W is closed under vector addition. Similarly, we must have a rule for multiplying any vector $\vec{w}$ in W by any scalar $r \in \mathbb{R}$ to produce a vector $r\vec{w}$ in W. Scalar multiplication in W is the same as in V, so it is necessary that W be closed under scalar multiplication. So if W is a subspace of V, then (1) and (2) are necessary.

Theorem 3.2 (continued)

Theorem 3.2. Test for Subspace.
A subset W of vector space V is a subspace if and only if
(1) $\vec{v}, \vec{w} \in W \Rightarrow \vec{v} + \vec{w} \in W$,
(2) for all $r \in \mathbb{R}$ and for all $\vec{v} \in W$ we have $r\vec{v} \in W$.

Proof (continued). Now suppose that W is nonempty and closed under vector addition and scalar multiplication (that is, (1) and (2) hold). If $\vec{0}$ is the only vector in W, then properties A1–A4 and S1–S4 are easily seen to hold since $\vec{v}, \vec{w} \in W$ implies $\vec{v} = \vec{w} = \vec{0}$. Then $W = \{\vec{0}\}$ is itself a vector space and so is a subspace of V. If nonzero vector $\vec{v}$ is in W then by closure under scalar multiplication, $(-1)\vec{v} = (-\vec{v}) \in W$. By closure under vector addition, $\vec{v} + (-\vec{v}) = \vec{0} \in W$. So $\vec{0} \in W$ and for any $\vec{v} \in W$ we have $-\vec{v} \in W$, as required of all vector spaces. Now A1–A4 and S1–S4 hold for all $\vec{v}, \vec{w} \in V$ and $r, s \in \mathbb{R}$, so they hold for all $\vec{v}, \vec{w} \in W$ and $r, s \in \mathbb{R}$. That is, W is itself a vector space and so is a subspace of V. $\square$

Page 202 Number 4

Page 202 Number 4. Determine whether the set F_1 of all functions f such that $f(1) = 0$ is a subspace of the vector space $\mathcal{F}$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3).

Solution. We apply Theorem 3.2, "Test for a Subspace." Let $f, g \in F_1$ and let $r \in \mathbb{R}$ be a scalar. Then $(f+g)(x) = f(x) + g(x)$, so $(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$ and so $f + g \in F_1$ and F_1 is closed under vector addition. Next, $(rf)(x) = rf(x)$, so $(rf)(1) = rf(1) = r0 = 0$ and so $rf \in F_1$ and F_1 is closed under scalar multiplication. So by Theorem 3.2, F_1 is a subspace of $\mathcal{F}$. $\square$

Page 202 Number 8

Page 202 Number 8. Let $\mathcal{P}$ be the vector space of polynomials with real coefficients along with the zero function (see Example 3.1.2). Prove that $\mathrm{sp}(1, x) = \mathrm{sp}(1 + 2x, x)$.

Proof. We show that each set of vectors is a subset of the other in order to deduce that the sets are the same.
Let $p(x) \in \mathrm{sp}(1, x)$. Then $p(x) = (r_1)1 + (r_2)x = r_1 + r_2x$ for some scalars $r_1, r_2 \in \mathbb{R}$. Now $p(x) = r_1 + r_2x = (r_1)(1 + 2x) + (r_2 - 2r_1)x$ and so $p(x) \in \mathrm{sp}(1 + 2x, x)$ (since $p(x)$ is a linear combination of $1 + 2x$ and x). Therefore every element of $\mathrm{sp}(1, x)$ is in $\mathrm{sp}(1 + 2x, x)$ and so $\mathrm{sp}(1, x) \subset \mathrm{sp}(1 + 2x, x)$.
Now let $q(x) \in \mathrm{sp}(1 + 2x, x)$. Then $q(x) = (s_1)(1 + 2x) + (s_2)x$ for some scalars $s_1, s_2 \in \mathbb{R}$. Now
$q(x) = (s_1)(1 + 2x) + (s_2)x = s_1 + 2s_1x + s_2x = (s_1)1 + (2s_1 + s_2)x$ and so $q(x) \in \mathrm{sp}(1, x)$. Therefore every element of $\mathrm{sp}(1 + 2x, x)$ is in $\mathrm{sp}(1, x)$ and so $\mathrm{sp}(1 + 2x, x) \subset \mathrm{sp}(1, x)$. Hence, $\mathrm{sp}(1, x) = \mathrm{sp}(1 + 2x, x)$. □

Page 202 Number 16

Page 202 Number 16. Determine whether the set of functions $\{\sin x, \sin 2x, \sin 3x\}$ is dependent or independent in the vector space $\mathcal{F}$ of all real-valued functions defined on $\mathbb{R}$ (see Example 3.1.3).

Solution. Suppose

$$r_1 \sin x + r_2 \sin 2x + r_3 \sin 3x = 0 \qquad (*)$$

for some scalars $r_1, r_2, r_3 \in \mathbb{R}$. Then this equation must hold for all $x \in \mathbb{R}$. In particular, for $x = \pi/2$ we have
$r_1 \sin(\pi/2) + r_2 \sin(2(\pi/2)) + r_3 \sin(3(\pi/2)) = 0$, or
$r_1(1) + r_2(0) + r_3(-1) = 0$ or

$$r_1 - r_3 = 0. \qquad (1)$$

Differentiating both sides of $(*)$ with respect to x implies that $r_1 \cos x + 2r_2 \cos 2x + 3r_3 \cos 3x = 0$ and with $x = 0$ we must have $r_1 \cos(0) + 2r_2 \cos(0) + 3r_2 \cos(0) = 0$ or

$$r_1 + 2r_2 + 3r_3 = 0. \qquad (2)$$

Page 202 Number 16 (continued 1)

Solution (continued). Taking a second derivative of $(*)$ with respect to x implies $-r_1 \sin x - 4r_2 \sin 2x - 9r_3 \sin 3x = 0$ and with $x = \pi/2$ we must have $-r_1 \sin(\pi/2) - 4r_2 \sin(2(\pi/2)) - 9r_3 \sin(3(\pi/2)) = 0$ or

$$-r_1 + 9r_3 = 0. \qquad (3)$$

So $(*)$ implies (1), (2), and (3) so that if $(*)$ holds then we must have
$\begin{array}{rcrcrcl} r_1 & & & - & r_3 & = & 0 \\ r_1 & + & 2r_2 & + & 3r_3 & = & 0 \\ -r_1 & & & + & 9r_3 & = & 0 \end{array}$. This system of equations has associated augmented matrix $\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 1 & 2 & 3 & 0 \\ -1 & 0 & 9 & 0 \end{array}\right]$. Since this is a homogeneous system of equations then any solution $[r_1, r_2, r_3]^T$ is a vector in the nullspace of the coefficient matrix A.

Page 202 Number 16 (continued 2)

Page 202 Number 16. Determine whether the set of functions $\{\sin x, \sin 2x, \sin 3x\}$ is dependent or independent in the vector space $\mathcal{F}$ of all real-valued functions defined on $\mathbb{R}$ (see Example 3.1.3).

Solution (continued). So we now reduce the coefficient matrix:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \\ -1 & 0 & 9 \end{bmatrix} \underset{R_3 \to R_3 + R_1}{\overset{R_2 \to R_2 - R_1}{\widetilde{}}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{bmatrix} = H.$$

Now H has 3 pivots and 0 pivot-free columns. So by Theorem 2.5(1), "The Rank Equation," the nullity of A is 0 and so the only solution to the system of equations is the trivial solution $r_1 = r_2 = r_3 = 0$. That is, the set of vectors is $\boxed{\text{linearly independent.}}$ □

Page 202 Number 20

Page 202 Number 20. Determine whether or not the set $\{x, x^2+1, (x-1)^2\}$ is a basis for the vector space $\mathcal{P}_2$ of all polynomials with real coefficients of degree 2 or less.

Solution. We use Definition 3.6, "Basis for a Vector Space," to see if the set is a linearly independent spanning set. For linear independence we consider the equation $(r_1)x + r_2(x^2+1) + r_3(x-1)^2 = 0x^2 + 0x + 0$. This gives $(r_2+r_3)x^2 + (r_1 - 2r_3)x + (r_2+r_3) = 0x^2 + 0x + 0$ and so we need $\begin{array}{ccccc} & r_2 & + & r_3 & = & 0 \\ r_1 & & - & 2r_3 & = & 0 \\ & r_2 & + & r_3 & = & 0 \end{array}$. We consider the augmented matrix for this system of equations:

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Page 202 Number 20 (continued)

Page 202 Number 20. Determine whether or not the set $\{x, x^2+1, (x-1)^2\}$ is a basis for the vector space $\mathcal{P}_2$ of all polynomials with real coefficients of degree 2 or less.

Solution (continued).

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \overset{R_3 \to R_3 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that the system of equations has a free variable, say $t = r_3$, and then the general solution is $r_1 = 2t$, $r_2 = -t$, $r_3 = t$. In particular, $r_1 = 2$, $r_2 = -1$, $r_3 = 1$ gives the dependence relation $(2)x + (-1)(x^2+1) + (1)(x-1)^2 = 0x^2 + 0x + 0$ and so, by Definition 3.5, "Linear Dependence and Independence," we see that the set $\{x, x^2+1, (x-1)^2\}$ is not linearly independent and so it $\boxed{\text{is not a basis for } \mathcal{P}_2.}$ □

Page 202 Number 22

Page 202 Number 22. Find a basis for $\text{sp}(x^2-1, x^2+1, 4, 2x-3)$ in the vector space $\mathcal{P}_2$ of all polynomials with real coefficients of degree 2 or less. (Notice that the text states this as a problem in $\mathcal{P}$ of all polynomials, but this does not affect our computations.)

Solution. Notice that $\dim(\mathcal{P}_2) = 3$ (see Note 3.2.C) and so there must be a dependence relation on the set of the 4 given vectors. So we consider $(r_1)(x^2-1) + (r_2)(x^2+1) + (r_3)4 + (r_4)(2x-3) = 0x^2 + 0x + 0$ or $(r_1+r_2)x^2 + (2r_4)x + (-r_1 + r_2 + 4r_3 - 3r_4) = 0x^2 + 0x + 0$. So we need

$$\begin{array}{ccccccccc} r_1 & + & r_2 & & & & & = & 0 \\ & & & & & & 2r_4 & = & 0 \\ -r_1 & + & r_2 & + & 4r_3 & - & 3r_4 & = & 0 \end{array}.$$

This system of equations yields the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ -1 & 1 & 4 & -3 & 0 \end{array}\right] \overset{R_3 \to R_3 + R_1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 4 & -3 & 0 \end{array}\right]$$

Page 202 Number 22 (continued 1)

Solution (continued).

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 4 & -3 & 0 \end{array}\right] \overset{R_2 \leftrightarrow R_3}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & -3 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \overset{R_1 \to R_1 - R_2/2}{\sim}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3/2 & 0 \\ 0 & 2 & 4 & -3 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \underset{R_2 \to R_2 + (3/2)R_3}{\overset{R_1 \to R_1 - (3/4)R_3}{\sim}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_1/2}{\overset{R_2 \to R_2/2}{\sim}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

With $t = r_3$ as a free variable we have $r_1 = 2t$, $r_2 = -2t$, $r_3 = t$, $r_4 = 0$. With $t = 1$ we see that $(2)(x^2-1) + (-2)(x^2+1) + (1)4 = 0$ or $4 = (-2)(x^2-1) + (2)(x^2+1)$. So 4 is a linear combination of $x^2 - 1$ and $x^2 + 1$. We remove it from the collection and consider the set $B = \{x^2-1, x^2+1, 2x-3\}$.

Page 202 Number 22 (continued 2)

Solution (continued). Set $B = \{x^2 - 1, x^2 + 1, 2x - 3\}$ is a linearly independent set since $r_1(x^2 - 1) + r_2(x^2 + 1) + r_3(2x - 3) = 0x^2 + 0x + 0$ implies $(r_1 + r_2)x^2 + (2r_3)x + (-r_1 + r_2 - 3r_3) = 0x^2 + 0x + 0$, or
$$\begin{array}{rcrcrcl} r_1 & + & r_2 & & & = & 0 \\ & & & & 2r_3 & = & 0 \\ -r_1 & + & r_2 & - & 3r_3 & = & 0 \end{array}$$
. This leads us to the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right] \overset{R_3 \to R_3 + R_1}{\sim} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right] \overset{R_2 \leftrightarrow R_3}{\sim} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\overset{R_1 \to R_1 - (1/2)R_2}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 3/2 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \overset{\substack{R_1 \to R_1 - (3/4)R_3 \\ R_2 \to R_2 + (3/2)R_3}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\overset{\substack{R_2 \to R_2/2 \\ R_3 \to R_3/2}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Page 202 Number 22 (continued 3)

Page 202 Number 22. Find a basis for $\text{sp}(x^2 - 1, x^2 + 1, 4, 2x - 3)$ in the vector space $\mathcal{P}_2$ of all polynomials with real coefficients of degree 2 or less. (Notice that the text states this as a problem in $\mathcal{P}$ of all polynomials, but this does not affect our computations.

Solution (continued). So we must have $r_1 = r_2 = r_3 = 0$ and hence the set $B = \{x^2 - 1, x^2 + 1, 2x - 3\}$ is linearly independent. We know set B to be a spanning set of $\text{sp}(x^2 - 1, x^2 + 1, 4, 2x - 3)$ since every linear combination of $x^2 - 1, x^2 + 1, 4, 2x - 3$ is also a linear combination of the elements of B (just replace the multiple of 4 with the same multiple of $(-2)(x^2 - 1) + (2)(x^2 + 1)$). Therefore, by Definition 3.6, "Basis for a Vector Space,"

$\boxed{B = \{x^2 - 1, x^2 + 1, 2x - 3\} \text{ is a basis for } \text{sp}(x^2 - 1, x^2 + 1, 4, 2x - 3).}$

□

Theorem 3.3

Theorem 3.3. Unique Combination Criterion for a Basis.
Let B be a set of nonzero vectors in vector space V. Then B is a basis for V if and only if each vector V can by uniquely expressed as a linear combination of the vectors in set B.

Proof. Suppose B is a basis for V. By Definition 3.6, "Basis for a Vector Space," B is a spanning set and so for any given $\vec{v} \in V$ there are $\vec{b}_1', \vec{b}_2', \ldots, \vec{b}_{k'}' \in B$ and $r_1', r_2', \ldots, r_{k'}' \in \mathbb{R}$ such that

$$\vec{v} = r_1'\vec{b}_1' + r_2'\vec{b}_2' + \cdots + r_{k'}'\vec{b}_{k'}'.$$

Suppose that $\vec{v}$ can be expressed as another linear combination of vectors, say

$$\vec{v} = s_1''\vec{b}_1'' + s_2''\vec{b}_2'' + \cdots + s_{k''}''\vec{b}_{k''}''$$

where $\vec{b}_1'', \vec{b}_2'', \ldots, \vec{b}_{k''}'' \in B$ and $s_1'', s_2'' \ldots, s_{k''}'' \in \mathbb{R}$. Some of the $\vec{b}_i'$ and $\vec{b}_i''$ may be the same or they could all be different. Let k be the number of different $\vec{b}_i'$ and $\vec{b}_i''$.

Theorem 3.3 (continued 1)

Proof (continued). Relabel these vectors as $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k$ and relabel the coefficients r_i' and s_i'' as r_i and s_i (introducing 0's as needed) such that $\vec{v} = r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_k\vec{b}_k = s_1\vec{b}_1 + s_2\vec{b}_2 + \cdots + s_k\vec{b}_k$ (this is necessary because the basis B might be infinite and so we cannot write $\vec{v}$ as an infinite linear combination; such things are not necessarily defined in a vector space and there are different levels of infinity, which complicates things further). Then we have

$$\vec{0} = \vec{v} - \vec{v} = (r_1 - s_1)\vec{b}_1 + (r_2 - s_2)\vec{b}_2 + \cdots + (r_k - s_k)\vec{b}_k$$

and since B is a basis then B is linearly independent (Definition 3.6) and so $r_1 - s_1 = r_2 - s_2 = \cdots = r_k - s_k = 0$ or $r_1 = s_1$, $r_2 = s_2$, ..., $r_k = s_k$. Therefore $\vec{v}$ can only be expressed in one way as a linear combination of elements of B, as claimed.

Theorem 3.3 (continued 2)

Theorem 3.3. Unique Combination Criterion for a Basis.
Let B be a set of nonzero vectors in vector space V. Then B is a basis for V if and only if each vector V can by uniquely expressed as a linear combination of the vectors in set B.

Proof (continued). Now suppose each vector $\vec{v}$ of V can be uniquely expressed as a linear combination of elements of B. Then B is a spanning set of V and (1) of Definition 3.6 holds. Let $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \in B$ and suppose that $r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_n\vec{b}_n = \vec{0}$. One choice for the coefficients $r_1, r_2, \ldots, r_n$ is $r_1 = r_2 = \cdots = r_n = 0$. But since $\vec{0}$ is a unique linear combination of $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$ then it is necessary that $r_1 = r_2 = \cdots = r_n = 0$. That is (by Definition 3.5, "Linear Dependence and Independence") B is linearly independent and (2) of Definition 3.6 holds. So Definition 3.6, "Basis for a Vector Space," is satisfied and B is a basis for V. □

Page 203 Number 32

Page 203 number 32. Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for V. If $\vec{w} \not\in \mathrm{sp}(\vec{v}_1, \vec{v}_2)$ then $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ is a basis for V.

Proof. By Definition 3.6, "Basis for a Vector Space," we need to show that $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ is a linearly independent spanning set of V. Since $\vec{w} \in V$, then $\vec{w} = r_1\vec{v}_1 + r_2\vec{v}_2 + r_3\vec{v}_3$ and $r_3 \neq 0$ since $\vec{w} \not\in \mathrm{sp}(\vec{v}_1, \vec{v}_2)$. Then $\vec{v}_3 = \dfrac{1}{r_3}(\vec{w} - r_1\vec{v}_1 - r_2\vec{v}_2)$. Therefore $\vec{v}_3 \in \mathrm{sp}(\vec{v}_1, \vec{v}_2, \vec{w})$. So

$$\mathrm{sp}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \subset \mathrm{sp}(\vec{v}_1, \vec{v}_2, \vec{w})$$

and so $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ generates V.

Next suppose, $s_1\vec{v}_1 + s_2\vec{v}_2 + s_3\vec{w} = \vec{0}$. Then $s_3 = 0$ or else $\vec{w} \in \mathrm{sp}(\vec{v}_1, \vec{v}_2)$. So $s_1\vec{v}_1 + s_2\vec{v}_2 = \vec{0}$ and so $s_1 = s_2 = 0$. Therefore $s_1 = s_2 = s_3 = 0$ and so $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ is a basis for V. □

Page 203 Number 36

Page 203 Number 36. Prove that if W is a subspace of an n-dimensional vector space V and $\dim(W) = n$, then $W = V$.

Proof. Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ be a basis of subspace W. ASSUME there is some $\vec{v} \in V$ such that $\vec{v} \not\in \mathrm{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$. Consider the equation

$$r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_n\vec{w}_n + r_{n+1}\vec{v} = \vec{0}. \qquad (*)$$

If $r_{n+1} \neq 0$ then $\vec{v} = -\dfrac{r_1}{r_{n+1}}\vec{w}_1 - \dfrac{r_2}{r_{n+1}}\vec{w}_2 - \cdots - \dfrac{r_n}{r_{n+1}}\vec{w}_n$ and $\vec{v} \in \mathrm{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$, in contradiction to the choice of $\vec{v}$. So $r_{n+1} = 0$ But then $(*)$ implies that $r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_n\vec{w}_n = \vec{0}$ and since $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ is a basis for W then by Definition 3.6, "Basis for a Vector Space," the vectors are linearly independent and so by Definition 3.5, "Linear Dependence and Independence," $r_1 = r_2 = \cdots = r_n = r_{n+1} = 0$ and so (by Definition 3.5) the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n, \vec{v}$ are $n+1$ linearly independent vectors.

Page 203 Number 36 (continued)

Page 203 Number 36. Prove that if W is a subspace of an n-dimensional vector space V and $\dim(W) = n$, then $W = V$.

Proof (continued). Then, since V is spanned by a set of n vectors (because it is dimension n), by Theorem 3.4, "Relative Size of Spanning and Independent Sets," $n \geq n+1$, a CONTRADICTION. So the assumption that there is $\vec{v} \in V$ such that $\vec{v} \not\in \mathrm{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$ is false. Hence $W = \mathrm{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$ includes all vectors in V and so $V \subset W$. Since W is a subspace of V then $W \subset V$ and therefore $V = W$, as claimed. □

Page 202 Number 40

Page 202 Number 40. A homogeneous linear nth-order differential equation has the form

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_2(x)y'' + f_1(x)y' + f_0(x)y = 0.$$

Prove that the set S of all solutions of this equation that lie in the vector space $\mathcal{F}$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3) is a subspace of $\mathcal{F}$.

Proof. We use Theorem 3.2, "Test for a Subspace." Let y_1 and y_2 be solutions of the differential equation and let $r \in \mathbb{R}$ be a scalar. Then consider $y = y_1 + y_2$ in the differential equation:

$$\begin{aligned} & f_n(x)(y_1+y_2)^{(n)} + f_{n-1}(x)(y_1+y_2)^{(n-1)} + \cdots \\ & + f_2(x)(y_1+y_2)'' + f_1(x)(y_1+y_2)' + f_0(x)(y_1+y_2) \\ = \; & f_n(x)\left(y_1^{(n)} + y_2^{(n)}\right) + f_{n-1}(x)\left(y_1^{(n-1)} + y_2^{(n-1)}\right) + \cdots \\ & + f_2(x)\left(y_1'' + y_2''\right) + f_1(x)\left(y_1' + y_2'\right) + f_0(x)(y_1+y_2) \ldots \end{aligned}$$

Page 202 Number 40 (continued 1)

Proof (continued).

$$\begin{aligned} & \text{since the derivative of a sum is the sum of the derivatives} \\ = \; & \left(f_n(x)y_1^{(n)} + f_{n-1}(x)y_1^{(n-1)} + \cdots + f_2(x)y_1'' + f_1(x)y_1' + f_0(x)y_1\right) \\ & + \left(f_n(x)y_2^{(n)} + f_{n-1}(x)y_2^{(n-1)} + \cdots + f_2(x)y_2'' + f_1(x)y_2' + f_0(x)y_2\right) \\ = \; & 0 + 0 \text{ since } y_1 \text{ and } y_2 \text{ are solutions to the differential equation} \\ = \; & 0. \end{aligned}$$

Therefore $y_1 + y_2$ is a solution to the differential equation and $y_1 + y_2 \in S$ and S is closed under vector addition.

Consider $y = ry_1$ in the differential equation:

$$\begin{aligned} & f_n(x)(ry_1)^{(n)} + f_{n-1}(x)(ry_1)^{(n-1)} + \cdots + f_2(x)(ry_1)'' + f_1(x)(ry_1)' + f_0(x)(ry_1) \\ = \; & f_n(x)ry_1^{(n)} + f_{n-1}(x)ry_1^{(n-1)} + \cdots + f_2(x)ry_1'' + f_1(x)ry_1' + f_0(x)ry_1 \\ & \text{since the derivative of a constant times a function is} \\ & \text{the constant times the derivative of the function} \end{aligned}$$

Page 202 Number 40 (continued 2)

Page 202 Number 40. A homogeneous linear nth-order differential equation has the form

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_2(x)y'' + f_1(x)y' + f_0(x)y = 0.$$

Prove that the set S of all solutions of this equation that lie in the vector space $\mathcal{F}$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3) is a subspace of $\mathcal{F}$.

Proof (continued).

$$\begin{aligned} = \; & r\left(f_n(x)y_1^{(n)} + f_{n-1}(x)y_1^{(n-1)} + \cdots + f_2(x)y_1'' + f_1(x)y_1' + f_0(x)y_1\right) \\ = \; & 0 \text{ since } y_1 \text{ is a solution to the differential equation.} \end{aligned}$$

So $ry_1 \in S$ and S is closed under scalar multiplication. Hence, by Theorem 3.2, S is a subspace of $\mathcal{F}$. □