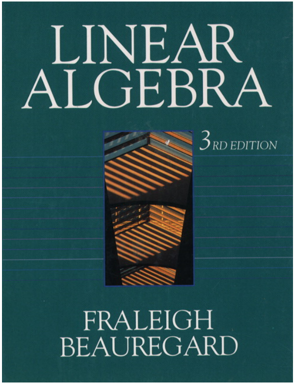
Linear Algebra

Chapter 3. Vector Spaces

Section 3.2. Basic Concepts of Vector Spaces—Proofs of Theorems



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Theorem 3.2

Theorem 3.2. Test for Subspace.
A subset W of vector space V is a subspace if and only if
(1) $\vec{v}, \vec{w} \in W \Rightarrow \vec{v} + \vec{w} \in W$,
(2) for all $r \in \mathbb{R}$ and for all $\vec{v} \in W$, we have $r\vec{v} \in W$.

Proof. Let W be a subspace of V. W must be nonempty since $\vec{0}$ must be in W by Definition 3.1, "Vector Space." Also by Definition 3.1, we see that W must have a rule for adding two vectors $\vec{v}$ and $\vec{w}$ in W to produce a vector $\vec{v} + \vec{w}$.

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Proof (continued). Now suppose that W is nonempty and closed under vector addition and scalar multiplication (that is, (1) and (2) hold). If $\vec{0}$ is the only vector in W, then properties A1–A4 and S1–S4 are easily seen to hold since $\vec{v}, \vec{w} \in W$ implies $\vec{v} = \vec{w} = \vec{0}$. Then $W = \{\vec{0}\}$ is itself a vector space and so is a subspace of V.

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Solution. We apply Theorem 3.2, "Test for a Subspace." Let $f, g \in F_1$ and let $r \in \mathbb{R}$ be a scalar.

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Page 202 Number 8. Let $\mathcal{P}$ be the vector space of polynomials with real coefficients along with the zero function (see Example 3.1.2). Prove that $\text{sp}(1, x) = \text{sp}(1 + 2x, x)$.

Proof. We show that each set of vectors is a subset of the other in order to deduce that the sets are the same.

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Let $p(x) \in \text{sp}(1, x)$. Then $p(x) = (r_1)1 + (r_2)x = r_1 + r_2x$ for some scalars $r_1, r_2 \in \mathbb{R}$. Now $p(x) = r_1 + r_2x = (r_1)(1 + 2x) + (r_2 - 2r_1)x$ and so $p(x) \in \text{sp}(1 + 2x, x)$ (since $p(x)$ is a linear combination of $1 + 2x$ and x). Therefore every element of $\text{sp}(1, x)$ is in $\text{sp}(1 + 2x, x)$ and so $\text{sp}(1, x) \subset \text{sp}(1 + 2x, x)$.

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Now let $q(x) \in \text{sp}(1 + 2x, x)$. Then $q(x) = (s_1)(1 + 2x) + (s_2)x$ for some scalars $s_1, s_2 \in \mathbb{R}$. Now $q(x) = (s_1)(1 + 2x) + (s_2)x = s_1 + 2s_1x + s_2x = (s_1)1 + (2s_1 + s_2)x$ and so $q(x) \in \text{sp}(1, x)$. Therefore every element of $\text{sp}(1 + 2x, x)$ is in $\text{sp}(1, x)$ and so $\text{sp}(1 + 2x, x) \subset \text{sp}(1, x)$.

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Now let $q(x) \in \text{sp}(1 + 2x, x)$. Then $q(x) = (s_1)(1 + 2x) + (s_2)x$ for some scalars $s_1, s_2 \in \mathbb{R}$. Now
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Page 202 Number 16. Determine whether the set of functions $\{\sin x, \sin 2x, \sin 3x\}$ is dependent or independent in the vector space $\mathcal{F}$ of all real-valued functions defined on $\mathbb{R}$ (see Example 3.1.3).

Solution. Suppose

$$r_1 \sin x + r_2 \sin 2x + r_3 \sin 3x = 0 \qquad (*)$$

for some scalars $r_1, r_2, r_3 \in \mathbb{R}$. Then this equation must hold for all $x \in \mathbb{R}$.

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for some scalars $r_1, r_2, r_3 \in \mathbb{R}$. Then this equation must hold for all $x \in \mathbb{R}$. In particular, for $x = \pi/2$ we have $r_1 \sin(\pi/2) + r_2 \sin(2(\pi/2)) + r_3 \sin(3(\pi/2)) = 0$, or $r_1(1) + r_2(0) + r_3(-1) = 0$ or

$$r_1 - r_3 = 0. \qquad (1)$$

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$$r_1 - r_3 = 0. \qquad (1)$$

Differentiating both sides of $(*)$ with respect to x implies that $r_1 \cos x + 2r_2 \cos 2x + 3r_3 \cos 3x = 0$ and with $x = 0$ we must have $r_1 \cos(0) + 2r_2 \cos(0) + 3r_2 \cos(0) = 0$ or

$$r_1 + 2r_2 + 3r_3 = 0. \qquad (2)$$

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Page 202 Number 16 (continued 1)

Solution (continued). Taking a second derivative of $(*)$ with respect to x implies $-r_1 \sin x - 4r_2 \sin 2x - 9r_3 \sin 3x = 0$ and with $x = \pi/2$ we must have $-r_1 \sin(\pi/2) - 4r_2 \sin(2(\pi/2)) - 9r_3 \sin(3(\pi/2)) = 0$ or

$$-r_1 + 9r_3 = 0. \qquad (3)$$

So $(*)$ implies (1), (2), and (3) so that if $(*)$ holds then we must have

$$\begin{array}{rcrcrcl} r_1 & & & - & r_3 & = & 0 \\ r_1 & + & 2r_2 & + & 3r_3 & = & 0 \\ -r_1 & & & + & 9r_3 & = & 0 \end{array}.$$

Page 202 Number 16 (continued 1)

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. This system of equations has associated augmented matrix $\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 1 & 2 & 3 & 0 \\ -1 & 0 & 9 & 0 \end{array}\right]$. Since this is a homogeneous system of equations then any solution $[r_1, r_2, r_3]^T$ is a vector in the nullspace of the coefficient matrix A.

Page 202 Number 16 (continued 1)

Solution (continued). Taking a second derivative of $(*)$ with respect to x implies $-r_1 \sin x - 4r_2 \sin 2x - 9r_3 \sin 3x = 0$ and with $x = \pi/2$ we must have $-r_1 \sin(\pi/2) - 4r_2 \sin(2(\pi/2)) - 9r_3 \sin(3(\pi/2)) = 0$ or

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Solution (continued). So we now reduce the coefficient matrix:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \\ -1 & 0 & 9 \end{bmatrix} \underset{R_3 \to R_3 + R_1}{\overset{R_2 \to R_2 - R_1}{\widetilde{}}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{bmatrix} = H.$$

Now H has 3 pivots and 0 pivot-free columns. So by Theorem 2.5(1), "The Rank Equation," the nullity of A is 0 and so the only solution to the system of equations is the trivial solution $r_1 = r_2 = r_3 = 0$. That is, the set of vectors is $\boxed{\text{linearly independent.}}$ $\square$

Page 202 Number 16 (continued 2)

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Solution (continued). So we now reduce the coefficient matrix:

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Now H has 3 pivots and 0 pivot-free columns. So by Theorem 2.5(1), "The Rank Equation," the nullity of A is 0 and so the only solution to the system of equations is the trivial solution $r_1 = r_2 = r_3 = 0$. That is, the set of vectors is $\boxed{\text{linearly independent.}}$ $\square$

Page 202 Number 20

Page 202 Number 20. Determine whether or not the set $\{x, x^2+1, (x-1)^2\}$ is a basis for the vector space $\mathcal{P}_2$ of all polynomials with real coefficients of degree 2 or less.

Solution. We use Definition 3.6, "Basis for a Vector Space," to see if the set is a linearly independent spanning set. For linear independence we consider the equation $(r_1)x + r_2(x^2+1) + r_3(x-1)^2 = 0x^2 + 0x + 0$. This gives $(r_2+r_3)x^2 + (r_1 - 2r_3)x + (r_2+r_3) = 0x^2 + 0x + 0$ and so we need

$$\begin{array}{rcrcrcl} & & r_2 & + & r_3 & = & 0 \\ r_1 & & & - & 2r_3 & = & 0 \\ & & r_2 & + & r_3 & = & 0 \end{array}.$$

Page 202 Number 20

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$$\begin{array}{ccccccc} & & r_2 & + & r_3 & = & 0 \\ r_1 & & & - & 2r_3 & = & 0 \\ & & r_2 & + & r_3 & = & 0 \end{array}.$$

We consider the augmented matrix for this system of equations:

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Page 202 Number 20

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$$\begin{array}{ccccccc} & & r_2 & + & r_3 & = & 0 \\ r_1 & & & - & 2r_3 & = & 0 \\ & & r_2 & + & r_3 & = & 0 \end{array}$$
. We consider the augmented matrix for this system of equations:

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \underbrace{\overrightarrow{R_1 \leftrightarrow R_2}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Page 202 Number 20 (continued)

Page 202 Number 20. Determine whether or not the set $\{x, x^2+1, (x-1)^2\}$ is a basis for the vector space $\mathcal{P}_2$ of all polynomials with real coefficients of degree 2 or less.

Solution (continued).

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \underset{\sim}{\overset{R_3 \to R_3 - R_2}{}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that the system of equations has a free variable, say $t = r_3$, and then the general solution is $r_1 = 2t$, $r_2 = -t$, $r_3 = t$. In particular, $r_1 = 2$, $r_2 = -1$, $r_3 = 1$ gives the dependence relation $(2)x + (-1)(x^2+1) + (1)(x-1)^2 = 0x^2 + 0x + 0$ and so, by Definition 3.5, "Linear Dependence and Independence," we see that the set $\{x, x^2+1, (x-1)^2\}$ is not linearly independent and so it

is not a basis for $\mathcal{P}_2$. □

Page 202 Number 20 (continued)

Page 202 Number 20. Determine whether or not the set $\{x, x^2+1, (x-1)^2\}$ is a basis for the vector space $\mathcal{P}_2$ of all polynomials with real coefficients of degree 2 or less.

Solution (continued).

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \underset{\sim}{\xrightarrow{R_3 \to R_3 - R_2}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that the system of equations has a free variable, say $t = r_3$, and then the general solution is $r_1 = 2t$, $r_2 = -t$, $r_3 = t$. In particular, $r_1 = 2$, $r_2 = -1$, $r_3 = 1$ gives the dependence relation $(2)x + (-1)(x^2+1) + (1)(x-1)^2 = 0x^2 + 0x + 0$ and so, by Definition 3.5, "Linear Dependence and Independence," we see that the set $\{x, x^2+1, (x-1)^2\}$ is not linearly independent and so it **is not a basis for $\mathcal{P}_2$.** □

Page 202 Number 22

Page 202 Number 22. Find a basis for $\mathrm{sp}(x^2 - 1, x^2 + 1, 4, 2x - 3)$ in the vector space $\mathcal{P}_2$ of all polynomials with real coefficients of degree 2 or less. (Notice that the text states this as a problem in $\mathcal{P}$ of all polynomials, but this does not affect our computations.)

Solution. Notice that $\dim(\mathcal{P}_2) = 3$ (see Note 3.2.C) and so there must be a dependence relation on the set of the 4 given vectors. So we consider $(r_1)(x^2 - 1) + (r_2)(x^2 + 1) + (r_3)4 + (r_4)(2x - 3) = 0x^2 + 0x + 0$ or $(r_1 + r_2)x^2 + (2r_4)x + (-r_1 + r_2 + 4r_3 - 3r_4) = 0x^2 + 0x + 0$.

Page 202 Number 22

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$$\begin{array}{rcrcrcrcl} r_1 & + & r_2 & & & & & = & 0 \\ & & & & & & 2r_4 & = & 0 \\ -r_1 & + & r_2 & + & 4r_3 & - & 3r_4 & = & 0 \end{array}.$$

Page 202 Number 22

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$$\begin{array}{rcrcrcrcl} r_1 & + & r_2 & & & & & = & 0 \\ & & & & & & 2r_4 & = & 0 \\ -r_1 & + & r_2 & + & 4r_3 & - & 3r_4 & = & 0 \end{array}.$$

This system of equations yields the augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ -1 & 1 & 4 & -3 & 0 \end{array}\right] \underset{\sim}{R_3 \to R_3 + R_1} \left[\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 4 & -3 & 0 \end{array}\right]$$

Page 202 Number 22

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Solution. Notice that $\dim(\mathcal{P}_2) = 3$ (see Note 3.2.C) and so there must be a dependence relation on the set of the 4 given vectors. So we consider $(r_1)(x^2-1)+(r_2)(x^2+1)+(r_3)4+(r_4)(2x-3) = 0x^2+0x+0$ or $(r_1+r_2)x^2+(2r_4)x+(-r_1+r_2+4r_3-3r_4) = 0x^2+0x+0$. So we need

$$\begin{array}{rcrcrcrcr} r_1 & + & r_2 & & & & & = & 0 \\ & & & & & & 2r_4 & = & 0 \\ -r_1 & + & r_2 & + & 4r_3 & - & 3r_4 & = & 0 \end{array}.$$

This system of equations yields the augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ -1 & 1 & 4 & -3 & 0 \end{array}\right] \underset{\sim}{\overset{R_3 \to R_3 + R_1}{}} \left[\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 4 & -3 & 0 \end{array}\right]$$

Page 202 Number 22 (continued 1)

Solution (continued).

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 4 & -3 & 0 \end{array}\right] \underset{\sim}{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & -3 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \underset{\sim}{R_1 \to R_1 - R_2/2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3/2 & 0 \\ 0 & 2 & 4 & -3 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \begin{array}{c} R_1 \to R_1 - (3/4)R_3 \\ \sim \\ R_2 \to R_2 + (3/2)R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 \to R_2/2 \\ \sim \\ R_3 \to R_1/2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

With $t = r_3$ as a free variable we have $r_1 = 2t$, $r_2 = -2t$, $r_3 = t$, $r_4 = 0$. With $t = 1$ we see that $(2)(x^2 - 1) + (-2)(x^2 + 1) + (1)4 = 0$ or $4 = (-2)(x^2 - 1) + (2)(x^2 + 1)$. So 4 is a linear combination of $x^2 - 1$ and $x^2 + 1$. We remove it from the collection and consider the set $B = \{x^2 - 1, x^2 + 1, 2x - 3\}$.

Page 202 Number 22 (continued 1)

Solution (continued).

$$\left[\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 4 & -3 & 0 \end{array}\right] \underbrace{\overset{R_2 \leftrightarrow R_3}{}}{} \left[\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & -3 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \underbrace{\overset{R_1 \to R_1 - R_2/2}{}}{}$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 3/2 & 0 \\ 0 & 2 & 4 & -3 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \underbrace{\overset{R_1 \to R_1 - (3/4)R_3}{}}_{R_2 \to R_2 + (3/2)R_3} \left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

$$\underbrace{\overset{R_2 \to R_2/2}{}}_{R_3 \to R_1/2} \left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

With $t = r_3$ as a free variable we have $r_1 = 2t$, $r_2 = -2t$, $r_3 = t$, $r_4 = 0$. With $t = 1$ we see that $(2)(x^2 - 1) + (-2)(x^2 + 1) + (1)4 = 0$ or $4 = (-2)(x^2 - 1) + (2)(x^2 + 1)$. So 4 is a linear combination of $x^2 - 1$ and $x^2 + 1$. We remove it from the collection and consider the set $B = \{x^2 - 1, x^2 + 1, 2x - 3\}$.

Page 202 Number 22 (continued 2)

Solution (continued). Set $B = \{x^2 - 1, x^2 + 1, 2x - 3\}$ is a linearly independent set since $r_1(x^2 - 1) + r_2(x^2 + 1) + r_3(2x - 3) = 0x^2 + 0x + 0$ implies $(r_1 + r_2)x^2 + (2r_3)x + (-r_1 + r_2 - 3r_3) = 0x^2 + 0x + 0$, or

$$\begin{array}{rcrcrcl} r_1 & + & r_2 & & & = & 0 \\ & & & & 2r_3 & = & 0 \\ -r_1 & + & r_2 & - & 3r_3 & = & 0 \end{array}$$

. This leads us to the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right] \underset{\sim}{\overset{R_3 \to R_3 + R_1}{}} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right] \underset{\sim}{\overset{R_2 \leftrightarrow R_3}{}} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\underset{\sim}{\overset{R_1 \to R_1 - (1/2)R_2}{}} \left[\begin{array}{rrr|r} 1 & 0 & 3/2 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \underset{R_2 \to R_2 + (3/2)R_3}{\overset{R_1 \to R_1 - (3/4)R_3}{\sim}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3/2}{\overset{R_2 \to R_2/2}{\sim}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Page 202 Number 22 (continued 2)

Solution (continued). Set $B = \{x^2 - 1, x^2 + 1, 2x - 3\}$ is a linearly independent set since $r_1(x^2 - 1) + r_2(x^2 + 1) + r_3(2x - 3) = 0x^2 + 0x + 0$ implies $(r_1 + r_2)x^2 + (2r_3)x + (-r_1 + r_2 - 3r_3) = 0x^2 + 0x + 0$, or

$$\begin{array}{rcrcrcl} r_1 & + & r_2 & & & = & 0 \\ & & & & 2r_3 & = & 0 \\ -r_1 & + & r_2 & - & 3r_3 & = & 0 \end{array}$$

. This leads us to the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right] \underset{\sim}{\overset{R_3 \to R_3 + R_1}{}} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right] \underset{\sim}{\overset{R_2 \leftrightarrow R_3}{}} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\underset{\sim}{\overset{R_1 \to R_1 - (1/2)R_2}{}} \left[\begin{array}{rrr|r} 1 & 0 & 3/2 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \begin{array}{c} R_1 \to R_1 - (3/4)R_3 \\ \sim \\ R_2 \to R_2 + (3/2)R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 \to R_2/2 \\ \sim \\ R_3 \to R_3/2 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Page 202 Number 22 (continued 3)

Page 202 Number 22. Find a basis for $\mathrm{sp}(x^2 - 1, x^2 + 1, 4, 2x - 3)$ in the vector space $\mathcal{P}_2$ of all polynomials with real coefficients of degree 2 or less. (Notice that the text states this as a problem in $\mathcal{P}$ of all polynomials, but this does not affect our computations.

Solution (continued). So we must have $r_1 = r_2 = r_3 = 0$ and hence the set $B = \{x^2 - 1, x^2 + 1, 2x - 3\}$ is linearly independent. We know set B to be a spanning set of $\mathrm{sp}(x^2 - 1, x^2 + 1, 4, 2x - 3)$ since every linear combination of $x^2 - 1, x^2 + 1, 4, 2x - 3$ is also a linear combination of the elements of B (just replace the multiple of 4 with the same multiple of $(-2)(x^2 - 1) + (2)(x^2 + 1)$). Therefore, by Definition 3.6, "Basis for a Vector Space,"

$B = \{x^2 - 1, x^2 + 1, 2x - 3\}$ is a basis for $\mathrm{sp}(x^2 - 1, x^2 + 1, 4, 2x - 3)$.

□

Page 202 Number 22 (continued 3)

Page 202 Number 22. Find a basis for $\text{sp}(x^2-1, x^2+1, 4, 2x-3)$ in the vector space $\mathcal{P}_2$ of all polynomials with real coefficients of degree 2 or less. (Notice that the text states this as a problem in $\mathcal{P}$ of all polynomials, but this does not affect our computations.

Solution (continued). So we must have $r_1 = r_2 = r_3 = 0$ and hence the set $B = \{x^2-1, x^2+1, 2x-3\}$ is linearly independent. We know set B to be a spanning set of $\text{sp}(x^2-1, x^2+1, 4, 2x-3)$ since every linear combination of $x^2-1, x^2+1, 4, 2x-3$ is also a linear combination of the elements of B (just replace the multiple of 4 with the same multiple of $(-2)(x^2-1)+(2)(x^2+1)$). Therefore, by Definition 3.6, "Basis for a Vector Space,"

$$\boxed{B = \{x^2-1, x^2+1, 2x-3\} \text{ is a basis for } \text{sp}(x^2-1, x^2+1, 4, 2x-3).}$$

□

Theorem 3.3

Theorem 3.3. Unique Combination Criterion for a Basis.
Let B be a set of nonzero vectors in vector space V. Then B is a basis for V if and only if each vector V can by uniquely expressed as a linear combination of the vectors in set B.

Proof. Suppose B is a basis for V. By Definition 3.6, "Basis for a Vector Space," B is a spanning set and so for any given $\vec{v} \in V$ there are $\vec{b}_1', \vec{b}_2', \ldots, \vec{b}_{k'}' \in B$ and $r_1', r_2', \ldots, r_{k'}' \in \mathbb{R}$ such that

$$\vec{v} = r_1'\vec{b}_1' + r_2'\vec{b}_2' + \cdots + r_{k'}'\vec{b}_{k'}'.$$

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$$\vec{v} = r'_1\vec{b}'_1 + r'_2\vec{b}'_2 + \cdots + r'_{k'}\vec{b}'_{k'}.$$

Suppose that $\vec{v}$ can be expressed as another linear combination of vectors, say

$$\vec{v} = s''_1\vec{b}''_1 + s''_2\vec{b}''_2 + \cdots + s''_{k''}\vec{b}''_{k''}$$

where $\vec{b}''_1, \vec{b}''_2, \ldots, \vec{b}''_{k''} \in B$ and $s''_1, s''_2 \ldots, s''_{k''} \in \mathbb{R}$.

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where $\vec{b}''_1, \vec{b}''_2, \ldots, \vec{b}''_{k''} \in B$ and $s''_1, s''_2 \ldots, s''_{k''} \in \mathbb{R}$. Some of the $\vec{b}'_i$ and $\vec{b}''_i$ may be the same or they could all be different. Let k be the number of different $\vec{b}'_i$ and $\vec{b}''_i$.

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Proof. Suppose B is a basis for V. By Definition 3.6, "Basis for a Vector Space," B is a spanning set and so for any given $\vec{v} \in V$ there are $\vec{b}_1', \vec{b}_2', \ldots, \vec{b}_{k'}' \in B$ and $r_1', r_2', \ldots, r_{k'}' \in \mathbb{R}$ such that

$$\vec{v} = r_1'\vec{b}_1' + r_2'\vec{b}_2' + \cdots + r_{k'}'\vec{b}_{k'}'.$$

Suppose that $\vec{v}$ can be expressed as another linear combination of vectors, say

$$\vec{v} = s_1''\vec{b}_1'' + s_2''\vec{b}_2'' + \cdots + s_{k''}''\vec{b}_{k''}''$$

where $\vec{b}_1'', \vec{b}_2'', \ldots, \vec{b}_{k''}'' \in B$ and $s_1'', s_2'' \ldots, s_{k''}'' \in \mathbb{R}$. Some of the $\vec{b}_i'$ and $\vec{b}_i''$ may be the same or they could all be different. Let k be the number of different $\vec{b}_i'$ and $\vec{b}_i''$.

Theorem 3.3 (continued 1)

Proof (continued). Relabel these vectors as $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k$ and relabel the coefficients r_i' and s_i'' as r_i and s_i (introducing 0's as needed) such that $\vec{v} = r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_k\vec{b}_k = s_1\vec{b}_1 + s_2\vec{b}_2 + \cdots + s_k\vec{b}_k$ (this is necessary because the basis B might be infinite and so we cannot write $\vec{v}$ as an infinite linear combination; such things are not necessarily defined in a vector space and there are different levels of infinity, which complicates things further). Then we have

$$\vec{0} = \vec{v} - \vec{v} = (r_1 - s_1)\vec{b}_1 + (r_2 - s_2)\vec{b}_2 + \cdots + (r_k - s_k)\vec{b}_k$$

and since B is a basis then B is linearly independent (Definition 3.6) and so $r_1 - s_1 = r_2 - s_2 = \cdots = r_k - s_k = 0$ or $r_1 = s_1$, $r_2 = s_2$, $\ldots$, $r_k = s_k$. Therefore $\vec{v}$ can only be expressed in one way as a linear combination of elements of B, as claimed.

Theorem 3.3 (continued 1)

Proof (continued). Relabel these vectors as $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k$ and relabel the coefficients r'_i and s''_i as r_i and s_i (introducing 0's as needed) such that $\vec{v} = r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_k\vec{b}_k = s_1\vec{b}_1 + s_2\vec{b}_2 + \cdots + s_k\vec{b}_k$ (this is necessary because the basis B might be infinite and so we cannot write $\vec{v}$ as an infinite linear combination; such things are not necessarily defined in a vector space and there are different levels of infinity, which complicates things further). Then we have

$$\vec{0} = \vec{v} - \vec{v} = (r_1 - s_1)\vec{b}_1 + (r_2 - s_2)\vec{b}_2 + \cdots + (r_k - s_k)\vec{b}_k$$

and since B is a basis then B is linearly independent (Definition 3.6) and so $r_1 - s_1 = r_2 - s_2 = \cdots = r_k - s_k = 0$ or $r_1 = s_1$, $r_2 = s_2$, $\ldots$, $r_k = s_k$. Therefore $\vec{v}$ can only be expressed in one way as a linear combination of elements of B, as claimed.

Theorem 3.3 (continued 2)

Theorem 3.3. Unique Combination Criterion for a Basis.
Let B be a set of nonzero vectors in vector space V. Then B is a basis for V if and only if each vector V can by uniquely expressed as a linear combination of the vectors in set B.

Proof (continued). Now suppose each vector $\vec{v}$ of V can be uniquely expressed as a linear combination of elements of B. Then B is a spanning set of V and (1) of Definition 3.6 holds. Let $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \in B$ and suppose that $r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_n\vec{b}_n = \vec{0}$. One choice for the coefficients $r_1, r_2, \ldots, r_n$ is $r_1 = r_2 = \cdots = r_n = 0$. But since $\vec{0}$ is a unique linear combination of $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$ then it is necessary that $r_1 = r_2 = \cdots = r_n = 0$. That is (by Definition 3.5, "Linear Dependence and Independence") B is linearly independent and (2) of Definition 3.6 holds.

Theorem 3.3 (continued 2)

Theorem 3.3. Unique Combination Criterion for a Basis.
Let B be a set of nonzero vectors in vector space V. Then B is a basis for V if and only if each vector V can by uniquely expressed as a linear combination of the vectors in set B.

Proof (continued). Now suppose each vector $\vec{v}$ of V can be uniquely expressed as a linear combination of elements of B. Then B is a spanning set of V and (1) of Definition 3.6 holds. Let $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n \in B$ and suppose that $r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_n\vec{b}_n = \vec{0}$. One choice for the coefficients $r_1, r_2, \dots, r_n$ is $r_1 = r_2 = \cdots = r_n = 0$. But since $\vec{0}$ is a unique linear combination of $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$ then it is necessary that $r_1 = r_2 = \cdots = r_n = 0$. That is (by Definition 3.5, "Linear Dependence and Independence") B is linearly independent and (2) of Definition 3.6 holds. So Definition 3.6, "Basis for a Vector Space," is satisfied and B is a basis for V. □

Theorem 3.3 (continued 2)

Theorem 3.3. Unique Combination Criterion for a Basis.
Let B be a set of nonzero vectors in vector space V. Then B is a basis for V if and only if each vector V can by uniquely expressed as a linear combination of the vectors in set B.

Proof (continued). Now suppose each vector $\vec{v}$ of V can be uniquely expressed as a linear combination of elements of B. Then B is a spanning set of V and (1) of Definition 3.6 holds. Let $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \in B$ and suppose that $r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_n\vec{b}_n = \vec{0}$. One choice for the coefficients $r_1, r_2, \ldots, r_n$ is $r_1 = r_2 = \cdots = r_n = 0$. But since $\vec{0}$ is a unique linear combination of $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$ then it is necessary that $r_1 = r_2 = \cdots = r_n = 0$. That is (by Definition 3.5, "Linear Dependence and Independence") B is linearly independent and (2) of Definition 3.6 holds. So Definition 3.6, "Basis for a Vector Space," is satisfied and B is a basis for V. □

Page 203 Number 32

Page 203 number 32. Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for V. If $\vec{w} \notin \text{sp}(\vec{v}_1, \vec{v}_2)$ then $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ is a basis for V.

Proof. By Definition 3.6, "Basis for a Vector Space," we need to show that $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ is a linearly independent spanning set of V. Since $\vec{w} \in V$, then $\vec{w} = r_1\vec{v}_1 + r_2\vec{v}_2 + r_3\vec{v}_3$ and $r_3 \neq 0$ since $\vec{w} \notin \text{sp}(\vec{v}_1, \vec{v}_2)$.

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$$\text{sp}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \subset \text{sp}(\vec{v}_1, \vec{v}_2, \vec{w})$$

and so $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ generates V.

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Next suppose, $s_1\vec{v}_1 + s_2\vec{v}_2 + s_3\vec{w} = \vec{0}$. Then $s_3 = 0$ or else $\vec{w} \in \mathrm{sp}(\vec{v}_1, \vec{v}_2)$. So $s_1\vec{v}_1 + s_2\vec{v}_2 = \vec{0}$ and so $s_1 = s_2 = 0$. Therefore $s_1 = s_2 = s_3 = 0$ and so $\{\vec{v}_1, \vec{v}_2, \vec{w}\}$ is a basis for V. □

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Page 203 Number 36

Page 203 Number 36. Prove that if W is a subspace of an n-dimensional vector space V and $\dim(W) = n$, then $W = V$.

Proof. Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ be a basis of subspace W.

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Proof. Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ be a basis of subspace W. ASSUME there is some $\vec{v} \in V$ such that $\vec{v} \notin \mathrm{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$. Consider the equation

$$r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_n\vec{w}_n + r_{n+1}\vec{v} = \vec{0}. \qquad (*)$$

If $r_{n+1} \neq 0$ then $\vec{v} = -\frac{r_1}{r_{n+1}}\vec{w}_1 - \frac{r_2}{r_{n+1}}\vec{w}_2 - \cdots - \frac{r_n}{r_{n+1}}\vec{w}_n$ and $\vec{v} \in \mathrm{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$, in contradiction to the choice of $\vec{v}$. So $r_{n+1} = 0$

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Page 203 Number 36 (continued)

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Proof (continued). Then, since V is spanned by a set of n vectors (because it is dimension n), by Theorem 3.4, "Relative Size of Spanning and Independent Sets," $n \geq n + 1$, a CONTRADICTION. So the assumption that there is $\vec{v} \in V$ such that $\vec{v} \notin \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$ is false. Hence $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$ includes all vectors in V and so $V \subset W$. Since W is a subspace of V then $W \subset V$ and therefore $V = W$, as claimed. □

Page 203 Number 36 (continued)

Page 203 Number 36. Prove that if W is a subspace of an n-dimensional vector space V and $\dim(W) = n$, then $W = V$.

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Page 202 Number 40

Page 202 Number 40. A homogeneous linear nth-order differential equation has the form

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_2(x)y'' + f_1(x)y' + f_0(x)y = 0.$$

Prove that the set S of all solutions of this equation that lie in the vector space $\mathcal{F}$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3) is a subspace of $\mathcal{F}$.

Proof. We use Theorem 3.2, "Test for a Subspace." Let y_1 and y_2 be solutions of the differential equation and let $r \in \mathbb{R}$ be a scalar.

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Proof. We use Theorem 3.2, "Test for a Subspace." Let y_1 and y_2 be solutions of the differential equation and let $r \in \mathbb{R}$ be a scalar. Then consider $y = y_1 + y_2$ in the differential equation:

$$f_n(x)(y_1 + y_2)^{(n)} + f_{n-1}(x)(y_1 + y_2)^{(n-1)} + \cdots$$
$$+ f_2(x)(y_1 + y_2)'' + f_1(x)(y_1 + y_2)' + f_0(x)(y_1 + y_2)$$
$$= f_n(x)\left(y_1^{(n)} + y_2^{(n)}\right) + f_{n-1}(x)\left(y_1^{(n-1)} + y_2^{(n-1)}\right) + \cdots$$
$$+ f_2(x)\left(y_1'' + y_2''\right) + f_1(x)\left(y_1' + y_2'\right) + f_0(x)(y_1 + y_2) \ldots$$

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$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_2(x)y'' + f_1(x)y' + f_0(x)y = 0.$$

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$$\begin{aligned} & f_n(x)(y_1 + y_2)^{(n)} + f_{n-1}(x)(y_1 + y_2)^{(n-1)} + \cdots \\ & +f_2(x)(y_1 + y_2)'' + f_1(x)(y_1 + y_2)' + f_0(x)(y_1 + y_2) \\ = \; & f_n(x)\left(y_1^{(n)} + y_2^{(n)}\right) + f_{n-1}(x)\left(y_1^{(n-1)} + y_2^{(n-1)}\right) + \cdots \\ & +f_2(x)\left(y_1'' + y_2''\right) + f_1(x)\left(y_1' + y_2'\right) + f_0(x)(y_1 + y_2) \ldots \end{aligned}$$

Page 202 Number 40 (continued 1)

Proof (continued).

$$\text{since the derivative of a sum is the sum of the derivatives}$$
$$= (f_n(x)y_1^{(n)} + f_{n-1}(x)y_1^{(n-1)} + \cdots + f_2(x)y_1'' + f_1(x)y_1' + f_0(x)y_1)$$
$$+(f_n(x)y_2^{(n)} + f_{n-1}(x)y_2^{(n-1)} + \cdots + f_2(x)y_2'' + f_1(x)y_2' + f_0(x)y_2)$$
$$= 0 + 0 \text{ since } y_1 \text{ and } y_2 \text{ are solutions to the differential equation}$$
$$= 0.$$

Therefore $y_1 + y_2$ is a solution to the differential equation and $y_1 + y_2 \in S$ and S is closed under vector addition.

Consider $y = ry_1$ in the differential equation:

$$f_n(x)(ry_1)^{(n)} + f_{n-1}(x)(ry_1)^{(n-1)} + \cdots + f_2(x)(ry_1)'' + f_1(x)(ry_1)' + f_0(x)(ry_1)$$

$$= f_n(x)ry_1^{(n)} + f_{n-1}(x)ry_1^{(n-1)} + \cdots + f_2(x)ry_1'' + f_1(x)ry_1' + f_0(x)ry_1$$

since the derivative of a constant times a function is the constant times the derivative of the function

Page 202 Number 40 (continued 1)

Proof (continued).

$$\begin{aligned}
&\quad \text{since the derivative of a sum is the sum of the derivatives} \\
&= (f_n(x)y_1^{(n)} + f_{n-1}(x)y_1^{(n-1)} + \cdots + f_2(x)y_1'' + f_1(x)y_1' + f_0(x)y_1) \\
&\quad +(f_n(x)y_2^{(n)} + f_{n-1}(x)y_2^{(n-1)} + \cdots + f_2(x)y_2'' + f_1(x)y_2' + f_0(x)y_2) \\
&= 0 + 0 \text{ since } y_1 \text{ and } y_2 \text{ are solutions to the differential equation} \\
&= 0.
\end{aligned}$$

Therefore $y_1 + y_2$ is a solution to the differential equation and $y_1 + y_2 \in S$ and S is closed under vector addition.

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$$\begin{aligned}
&= f_n(x)ry_1^{(n)} + f_{n-1}(x)ry_1^{(n-1)} + \cdots + f_2(x)ry_1'' + f_1(x)ry_1' + f_0(x)ry_1 \\
&\quad \text{since the derivative of a constant times a function is} \\
&\quad \text{the constant times the derivative of the function}
\end{aligned}$$

Page 202 Number 40 (continued 2)

Page 202 Number 40. A homogeneous linear *n*th-order differential equation has the form

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_2(x)y'' + f_1(x)y' + f_0(x)y = 0.$$

Prove that the set S of all solutions of this equation that lie in the vector space $\mathcal{F}$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3) is a subspace of $\mathcal{F}$.

Proof (continued).

$$\begin{aligned} &= r\left(f_n(x)y_1^{(n)} + f_{n-1}(x)y_1^{(n-1)} + \cdots + f_2(x)y_1'' + f_1(x)y_1' + f_0(x)y_1\right) \\ &= 0 \text{ since } y_1 \text{ is a solution to the differential equation.} \end{aligned}$$

So $ry_1 \in S$ and S is closed under scalar multiplication. Hence, by Theorem 3.2, S is a subspace of $\mathcal{F}$. □

Page 202 Number 40 (continued 2)

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$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_2(x)y'' + f_1(x)y' + f_0(x)y = 0.$$

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Proof (continued).

$$\begin{aligned} &= r\left(f_n(x)y_1^{(n)} + f_{n-1}(x)y_1^{(n-1)} + \cdots + f_2(x)y_1'' + f_1(x)y_1' + f_0(x)y_1\right) \\ &= 0 \text{ since } y_1 \text{ is a solution to the differential equation.} \end{aligned}$$

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