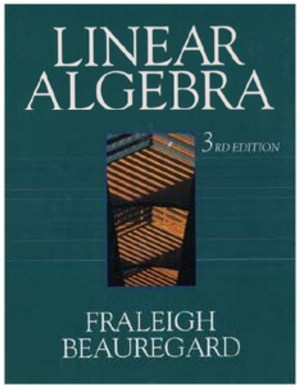
Linear Algebra

Chapter 3. Vector Spaces

Section 3.3. Coordinatization of Vectors—Proofs of Theorems



Page 211 Number 6

Page 211 Number 6. Find the coordinate vector of $\vec{v} = [9, 6, 11, 0] \in \mathbb{R}^4$ relative to the ordered basis
$B = ([1, 0, 1, 0], [2, 1, 1, -1], [0, 1, 1, -1], [2, 1, 3, 1])$.

Solution. By Definition 3.8, "Coordinate Vector Relative to an Ordered Basis," we need to find scalars r_1, r_2, r_3, r_4 such that

$$r_1[1, 0, 1, 0] + r_2[2, 1, 1, -1] + r_3[0, 1, 1, -1] + r_4[2, 1, 3, 1] = [9, 6, 11, 0];$$

that is, we need
$[r_1 + 2r_2 + 2r_4, r_2 + r_3 + r_4, r_1 + r_2 + r_3 + 3r_4, -r_2 - r_3 + r_4] = [9, 6, 11, 0]$.
So we need

$$\begin{array}{ccccccccc} r_1 & + & 2r_2 & & & + & 2r_4 & = & 9 \\ & & r_2 & + & r_3 & + & r_4 & = & 6 \\ r_1 & + & r_2 & + & r_3 & + & 3r_4 & = & 11 \\ & - & r_2 & - & r_3 & + & r_4 & = & 0. \end{array}$$

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Solution (continued). We consider the augmented matrix for the system of equations and now reduce it:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 9 \\ 0 & 1 & 1 & 1 & 6 \\ 1 & 1 & 1 & 3 & 11 \\ 0 & -1 & -1 & 1 & 0 \end{array}\right] \underset{R_3 \to R_3 - R_1}{\sim} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 9 \\ 0 & 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} R_1 \to R_1 - 2R_2 \\ \underset{R_3 \to R_3 + R_2}{\sim} \\ R_4 \to R_4 + R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -3 \\ 0 & 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 2 & 8 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right] \begin{array}{c} \underset{R_3 \to R_3/2}{\sim} \\ R_4 \to R_4/2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -3 \\ 0 & 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

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Solution (continued).

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -3 \\ 0 & 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} \underset{R_1 \to R_1 + 2R_3}{\sim} \\ R_2 \to R_2 - R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} \underset{R_1 \to R_1 - 2R_4}{\sim} \\ R_3 \to R_3 - R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right].$$

So we need $r_1 = -1$, $r_2 = 2$, $r_3 = 1$, and $r_4 = 3$. Hence
$\boxed{\vec{v}_B = [-1, 2, 1, 3].}$ □

Theorem 3.3.A. The Fundamental Theorem of Finite Dimensional Vectors Spaces

Theorem 3.3.A. The Fundamental Theorem of Finite Dimensional Vectors Spaces. If V is a finite dimensional vector space (say $\dim(V) = n$) then V is isomorphic to $\mathbb{R}^n$.

Proof. Let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ be an ordered basis for V and for $\vec{v} \in V$ with $\vec{v}_B = [r_1, r_2, \ldots, r_n]$ define $\alpha : V \to \mathbb{R}^n$ as $\alpha(\vec{v}) = [r_1, r_2, \ldots, r_n] = \vec{v}_B$. Then "clearly" α is one-to-one and onto. Also for $\vec{v}, \vec{w} \in V$ suppose $\vec{v}_B = [r_1, r_2, \ldots, r_n]$ and $\vec{w}_B = [s_1, s_2, \ldots, s_n]$ so that $(\vec{v} + \vec{w})_B = \left((r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_n\vec{b}_n) + (s_1\vec{b}_1 + s_2\vec{b}_2 + \cdots + s_n\vec{b}_n)\right)_B =$ $\left((r_1 + s_1)\vec{b}_1 + (r_2 + s_2)\vec{b}_2 + \cdots + (r_n + s_n)\vec{b}_n\right)_B =$ $[r_1 + s_1, r_2 + s_2, \ldots, r_n + s_n]$ and so

$$\begin{aligned} \alpha(\vec{v} + \vec{w}) &= [r_1 + s_1, r_2 + s_2, \ldots, r_n + s_n] \\ &= [r_1, r_2, \ldots, r_n] + [s_1, s_2, \ldots, s_n] = \alpha(\vec{v}) + \alpha(\vec{w}). \end{aligned}$$

Theorem 3.3.A. The Fundamental Theorem of Finite Dimensional Vectors Spaces (continued)

Theorem 3.3.A. The Fundamental Theorem of Finite Dimensional Vectors Spaces.
If V is a finite dimensional vector space (say $\dim(V) = n$) then V is isomorphic to $\mathbb{R}^n$.

Proof (continued). For a scalar $t \in \mathbb{R}$, $t\vec{v} = t(r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_n\vec{b}_n) = (tr_1)\vec{b}_1 + (tr_2)\vec{b}_2 + \cdots + (tr_n)\vec{b}_n$ and so $(t\vec{v})_B = [tr_1, tr_2, \ldots, tr_n]$. Hence

$$\alpha(t\vec{v}) = [tr_1, tr_2, \ldots, tr_n] = t[r_1, r_2, \ldots, r_n] = t\alpha(\vec{v}).$$

So α is an isomorphism and V is isomorphic to $\mathbb{R}^n$. □

Example 3.3.B. Isomorphism of $\mathcal{P}_n$

Example. Consider $\mathcal{P}_n$, the vector space of all polynomials of degree n or less (see Exercise 3.1.16). Since $\dim(\mathcal{P}_n) = n + 1$ (see Section 3.2), so $\mathcal{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$. Find an isomorphism and prove that it is an isomorphism.

Proof. For $p(x) \in \mathcal{P}_n$, say $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, define $\alpha : \mathcal{P}_n \to \mathbb{R}^{n+1}$ as $\alpha(p(x)) = [a_n, a_{n-1}, \ldots, a_1, a_0]$. Clearly α is one to one (each vector in $\mathbb{R}^{n+1}$ is the image of only one polynomial in $\mathcal{P}_n$) and onto (each vector in $\mathbb{R}^{n+1}$ is the image of some polynomial in $\mathcal{P}_n$). To show that α is an isomorphism we consider $p(x), q(x) \in \mathcal{P}_n$ (say $q(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$) and scalar $r \in \mathbb{R}$. Then

$$\begin{aligned} \alpha(p(x) + q(x)) &= \alpha\left((a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)\right. \\ &\quad \left.+(b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0)\right) \\ &= \alpha((a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots \\ &\quad +(a_1 + b_1)x + (a_0 + b_0)) \end{aligned}$$

Example 3.3.B. Isomorphism of $\mathcal{P}_n$ (continued)

Proof (continued).

$$\begin{aligned} \alpha(p(x) + q(x)) &= \alpha((a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots \\ &\quad +(a_1 + b_1)x + (a_0 + b_0)) \\ &= [a_n + b_n, a_{n-1} + b_{n-1}, \ldots, a_1 + b_1, a_0 + b_0] \\ &= [a_n, a_{n-1}, \ldots, a_1, a_0] + [b_n, b_{n-1}, \ldots, b_0] \\ &= \alpha(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) \\ &\quad +\alpha(b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0) \\ &= \alpha(p(x)) + \alpha(q(x)). \end{aligned}$$

$$\begin{aligned} \text{Also, } \alpha(rp(x)) &= \alpha\left(r(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)\right) \\ &= \alpha\left((ra_n)x^n + (ra_{n-1})x^{n-1} + \cdots + (ra_1)x + (ra_0)\right) \\ &= [ra_n, ra_{n-1}, \ldots, ra_1, ra_0] = r[a_n, a_{n-1}, \ldots, a_1, a_0] \\ &= r\alpha(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) = r\alpha(p(x)). \end{aligned}$$

So α satisfies the definition of an isomorphism, as required. □

Page 212 Number 12

Page 212 Number 12. Find the coordinate vector of the polynomial $p(x) = 4x^3 - 9x^2 + x$ relative to the ordered basis $B' = ((x-1)^3, (x-1)^2, (x-1), 1)$ of the vector space $\mathcal{P}_3$ of polynomials of degree 3 or less.

Solution. We express each basis vector in B' as a coordinate vector relative to the basis $B = (x^3, x^2, x, 1)$ of $\mathcal{P}_3$, so that

$$\begin{aligned} (x-1)^3_B &= (x^3 - 3x^2 + 3x - 1)_B = [1, -3, 3, -1] = \vec{b}_1 \\ (x-1)^2_B &= (x^2 - 2x + 1)_B = [0, 1, -2, 1] = \vec{b}_2 \\ (x-1)_B &= [0, 0, 1, -1] = \vec{b}_3, \text{ and} \\ 1_B &= [0, 0, 0, 1] = \vec{b}_4. \end{aligned}$$

Also, $p(x)_B = [4, -9, 1, 0]$.

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Solution. Now we use techniques of Note 3.3.A and we consider

$$[\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4 \mid p(x)_B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ -3 & 1 & 0 & 0 & -9 \\ 3 & -2 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} R_2 \to R_2 + 3R_1 \\ \underset{\sim}{R_3 \to R_3 - 3R_1} \\ R_4 \to R_4 + R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & -2 & 1 & 0 & -11 \\ 0 & 1 & -1 & 1 & 4 \end{array}\right] \begin{array}{c} \underset{\sim}{R_3 \to R_3 + 2R_2} \\ R_4 \to R_4 - R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

$$\underset{\sim}{R_4 \to R_4 + R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right].$$

So $[4, 3, -5, -4] \in \mathbb{R}^4$ is the representation of p with respect to basis B'; $\boxed{p(x)_{B'} = [4, 3, -5, -4].}$ □

Page 212 Number 20

Page 212 Number 20. Prove the set $\{(x-a)^n, (x-a)^{n-1}, \ldots, (x-a), 1\}$ is a basis for $\mathcal{P}_n$.

Proof. Let $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{n+1}$ be the coordinate vectors of $(x-a)^n, (x-a)^{n-1}, \ldots, (x-a), 1$ in terms of the ordered basis $(x^n, x^{n-1}, \ldots, x^2, x, 1)$. Form a matrix A with the $\vec{v}_k$s as the columns:

$$A = [\vec{v}_0\ \vec{v}_1\ \cdots\ \vec{v}_n].$$

By the Binomial Theorem,

$$(x-a)^k = \sum_{i=0}^{k} \binom{k}{i} x^{k-i}(-a)^i \text{ where } \binom{k}{i} = \frac{k!}{(k-i)!i!}.$$

So $\vec{v}_{n-k} =$
$\left[0, 0, \ldots, 0, 1, -ka, \frac{k(k-1)}{2}a^2, \ldots, \frac{k(k-1)}{2}(-a)^{k-2}, k(-a)^{k-1}, (-a)^k\right]$,
where the first $n-k$ components of $\vec{v}_k$ are 0.

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Solution (continued). Notice that A is "lower triangular":

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -na & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{n(n-1)}{2}a^2 & -(n-1)a & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{n(n-1)}{2}(-a)^{n-2} & \frac{(n-1)(n-2)}{2}(-a)^{n-3} & \frac{(n-2)(n-3)}{2}(-a)^{n-4} & \cdots & 1 & 0 & 0 \\ n(-a)^{n-1} & (n-1)(-a)^{n-2} & (n-2)(-a)^{n-3} & \cdots & -2 & 1 & 0 \\ (-a)^n & (-a)^{n-1} & (-a)^{n-2} & \cdots & a^2 & -a & 1 \end{bmatrix}$$

Now the rank of A and the rank of A^T are the same (by Theorem 2.4, say). A^T is "upper triangular" and so has rank equal to the number of columns $n+1$. So the rank of A is $n+1$, A is row equivalent to the identity and so the $\vec{v}_k$ are linearly independent. Since $\dim(\mathcal{P}_n) = n+1$ and the set

$$\{(x-a)^n, (x-a)^{n-1}, \ldots, (x-a), 1\}$$

is a set of $n+1$ linearly independent vectors, then this set is a basis for $\mathcal{P}_n$. □