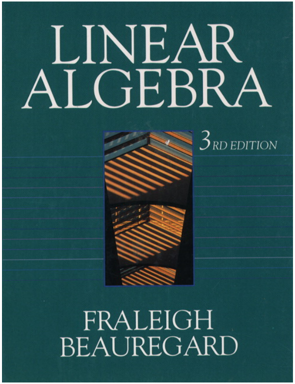
# Linear Algebra

**Chapter 3. Vector Spaces**

Section 3.4. Linear Transformations—Proofs of Theorems



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# Example 3.4.A

**Example 3.4.A.** Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$. Is $T$ a linear transformation?

**Solution.** We use Note 3.4.A. Let $f, g \in \mathcal{F}$ and let $r, s \in \mathbb{R}$. Then

$$\begin{aligned} T(rf + sg) &= a(rf + sg) \\ &= a(rf) + a(sg) \text{ by S1} \\ &= (ar)f + (as)g \text{ by S3} \\ &= (ra)f + (sa)g \text{ by commutivity in } \mathbb{R} \\ &= r(af) + s(ag) \text{ by S3} \\ &= rT(f) + sT(g). \end{aligned}$$

Therefore, $\boxed{\text{yes, } T \text{ is a linear transformation.}}$ □

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**Solution.** Let $f \in \ker(T)$. Then $T(f) = 0$ (where $0 = 0(x)$ denotes the constant function which is 0 for all $x \in \mathbb{R}$). So $T(f) = af = af(x) = 0(x) = 0$. Since $a \neq 0$ then $f(x) = 0$ for all $x \in \mathbb{R}$. That is, $f(x) = 0(x)$ or $f = 0$. So $\boxed{\ker(T) = \{0\} = \{0(x)\}.}$ □

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**Page 214 Example 1.** Let $\mathcal{F}$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ (see Example 3.1.3), and let $D$ be its subspace of all differentiable functions. Show that differentiation is a linear transformation of $D$ into $F$.

**Proof.** Let $T : D \to F$ be defined as $T(f) = f'$. Let $f, g \in D$ and let $r \in \mathbb{R}$. Since the derivative of a sum is the sum of the derivatives, then

$$T(f+g) = (f+g)' = f' + g' = T(f) + T(g).$$

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# Page 215 Example 3

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**Proof.** Let $f, g \in C_{a,b}$ and let $r \in \mathbb{R}$ be a scalar. Since the integral of a sum is the sum of the integrals and the integral of a multiple of a function is the multiple of the integral of the function, we have

$$T(f+g) = \int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = T(f)+T(g)$$

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**Proof.** Similar to the previous example, for $f, g \in C$ and for scalar $r \in \mathbb{R}$ we have

$$T_a(f+g) = \int_a^x (f(t)+g(t))\,dt = \int_a^x f(t)\,dt + \int_a^x g(t)\,dt = T_a(f) + T_a(g)$$

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So by Definition 3.9, "Linear Transformation," $T_a$ is a linear transformation. □

# Theorem 3.5

**Theorem 3.5. Preservation of Zero and Subtraction**
Let $V$ and $V'$ be vectors spaces, and let $T : V \to V'$ be a linear transformation. Then
**(1)** $T(\vec{0}) = \vec{0}'$, and
**(2)** $T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2)$, for any vectors $\vec{v}_1$ and $\vec{v}_2$ in $V$.

**Proof.** First,

$$\begin{aligned} T(\vec{0}) &= T(0\vec{0}) \text{ by Theorem 3.1(4),} \\ &\quad \text{"Elementary Properties of Vector Spaces"} \\ &= 0T(\vec{0}) \text{ by Definition 3.9(2),} \\ &\quad \text{"Linear Transformations"} \\ &= \vec{0}' \text{ by Theorem 3.1(4).} \end{aligned}$$

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# Theorem 3.5 (continued)

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Let $V$ and $V'$ be vectors spaces, and let $T : V \to V'$ be a linear transformation. Then
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**Proof (continued).** Second,

$$\begin{aligned} T(\vec{v}_1 - \vec{v}_2) &= T(\vec{v}_1 - (1)\vec{v}_2) \text{ by S4} \\ &= T(\vec{v}_1 + (-1)\vec{v}_2) \text{ by Theorem 3.1(6)} \\ &= T(\vec{v}_1) + (-1)T(\vec{v}_2) \text{ by Note 3.4.A} \\ &= T(\vec{v}_1) - T(\vec{v}_2) \text{ by Theorem 3.1(6).} \end{aligned}$$

So (1) and (2) hold, as claimed. □

# Theorem 3.6

**Theorem 3.6. Bases and Linear Transformations.**
Let $T : V \to V'$ be a linear transformation, and let $B$ be a basis for $V$. For any vector $\vec{v}$ in $V$, the vector $T(\vec{v})$ is uniquely determined by the vectors $T(\vec{b})$ for all $\vec{b} \in B$.

**Proof.** Let $T$ and $\overline{T}$ be two linear transformations such that $T(\vec{b}_i) = \overline{T}(\vec{b}_i)$ for each vector $\vec{b}_i \in B$. Let $\vec{v} \in V$. Then for some scalars $r_1, r_2, \ldots, r_k$ we have $\vec{v} = r_1\vec{b}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{b}_k$.

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$$\begin{aligned} T(\vec{v}) &= T(r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_k\vec{b}_k) \\ &= r_1 T(\vec{b}_1) + r_2 T(\vec{b}_2) + \cdots + r_k T(\vec{b}_k) \text{ by Note 3.4.A} \\ &= r_1 \overline{T}(\vec{b}_1) + r_2 \overline{T}(\vec{b}_2) + \cdots + r_k \overline{T}(\vec{b}_k) \\ &= \overline{T}(r_1\vec{b}_1 + r_2\vec{b}_2 + \cdots + r_k\vec{b}_k) \text{ by Note 3.4.A} \\ &= \overline{T}(\vec{v}). \end{aligned}$$

Therefore $T$ and $\overline{T}$ are the same transformations. □

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# Theorem 3.4.A

**Theorem 3.4.A.** (Page 229 number 46) Let $T : V \to V'$ be a linear transformation and let $T(\vec{p}) = \vec{b}$ for a particular vector $\vec{p}$ in $V$. The solution set of $T(\vec{x}) = \vec{b}$ is the set $\{\vec{p} + \vec{h} \mid \vec{h} \in \ker(T)\}$.

**Proof.** Let $\vec{p}$ be a solution of $T(\vec{v}) = \vec{b}$. Then $T(\vec{p}) = \vec{b}$. Let $\vec{h}$ be a solution of $T(\vec{x}) = \vec{0'}$. Then $T(\vec{h}) = \vec{0'}$.

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$$T(\vec{p} + \vec{h}) = T(\vec{p}) + T(\vec{h}) = \vec{b} + \vec{0'} = \vec{b},$$

and so $\vec{p} + \vec{h}$ is indeed a solution. Also, if $\vec{q}$ is any solution of $T(\vec{x}) = \vec{b}$ then by Theorem 3.5(2), "Preservation of Zero and Subtraction,"

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and so $\vec{q} - \vec{p}$ is in the kernel of $T$. Therefore for some $\vec{h} \in \ker(T)$, we have $\vec{q} - \vec{p} = \vec{h}$, for $\vec{q} = \vec{p} + \vec{h}$. □

# Corollary 3.4.A

**Corollary 3.4.A. One-to-One and Kernel.**
A linear transformation $T$ is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

**Proof.** Let $T : V \to V'$ where $V$ and $V'$ are vector spaces.

Let $\ker(T) = \{\vec{0}\}$. Suppose for some $\vec{v}_1, \vec{v}_2 \in V$ we have $T(\vec{v}_1) = T(\vec{v}_2)$. Then $T(\vec{v}_1) - T(\vec{v}_2) = \vec{0}'$ and so by Theorem 3.5(2), Preservation of Zero and Subtraction, $T(\vec{v}_1 - \vec{v}_2) = \vec{0}'$. That is, $\vec{v}_1 - \vec{v}_2 \in \ker(T) = \{\vec{0}\}$. So it must be that $\vec{v}_1 - \vec{v}_2 = \vec{0}$, or $\vec{v}_1 = \vec{v}_2$, and hence $T$ is one-to-one.

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Next, suppose $T$ is one-to-one. Since $T(\vec{0}) = \vec{0}'$ by Theorem 3.5(1), "Preservation of Zero and Subtraction," then for any nonzero vector $\vec{x} \in V$ we must have that $T(\vec{x}) \neq \vec{0}'$. That is, the only vector in $\ker(T)$ is $\vec{0}$. So $\ker(T) = \{\vec{0}\}$, as claimed. □

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# Theorem 3.8

**Theorem 3.8.** A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

**Proof.** ASSUME $T$ is invertible and is not one-to-one. Then by the definition of "one-to-one," for some $\vec{v}_1 \neq \vec{v}_2$ both in $V$, we have $T(\vec{v}_1) = T(\vec{v}_2) = \vec{v}'$.

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From Definition 3.10, "Invertible Transformation," if $T$ is invertible then for any $\vec{v}' \in V'$ we must have $T^{-1}(\vec{v}') = \vec{v}$ for some $\vec{v} \in V$. Therefore the image of $\vec{v}$ is $\vec{v}' \in V'$ and $T$ is onto.

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# Theorem 3.8 (continued 1)

**Theorem 3.8.** A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

**Proof (continued).** Finally, we need to show that if $T$ is one-to-one and onto then it is invertible. Suppose that $T$ is one-to-one and onto $V'$. Since $T$ is onto $V'$, then for each $\vec{v}' \in V'$ we can find $\vec{v} \in V$ such that $T(\vec{v}) = \vec{v}'$ and because $T$ is one-to-one, this vector $\vec{v} \in V$ is unique (from the definition of "one-to-one" and "onto").

# Theorem 3.8 (continued 1)

**Theorem 3.8.** A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

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$$(T \circ T^{-1})(\vec{v}') = T(T^{-1}(\vec{v}')) = T(\vec{v}) = \vec{v}'$$

and

$$(T^{-1} \circ T)(\vec{v}) = T^{-1}(T(\vec{v})) = T^{-1}(\vec{v}') = \vec{v},$$

and so $T \circ T^{-1}$ is the identity map on $V'$ and $T^{-1} \circ T$ is the identity map on $V$.

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# Theorem 3.8 (continued 2)

**Theorem 3.8.** A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

**Proof (continued).** Now we need only show that $T^{-1}$ is linear. Suppose $T(\vec{v}_1) = \vec{v}_1'$ and $T(\vec{v}_2) = \vec{v}_2'$; that is, $\vec{v}_1 = T^{-1}(\vec{v}_1')$ and $\vec{v}_2 = T^{-1}(\vec{v}_2')$.

# Theorem 3.8 (continued 2)

**Theorem 3.8.** A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

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$$\begin{aligned} T^{-1}(\vec{v}_1' + \vec{v}_2') &= T^{-1}(T(\vec{v}_1) + T(\vec{v}_2)) \\ &= T^{-1}(T(\vec{v}_1 + \vec{v}_2)) \text{ since } T \text{ is linear} \\ &= (T^{-1} \circ T)(\vec{v}_1 + \vec{v}_2) = \mathcal{I}(\vec{v}_1 + \vec{v}_2) = \vec{v}_1 + \vec{v}_2 \\ &= T^{-1}(\vec{v}_1') + T^{-1}(\vec{v}_2'). \end{aligned}$$

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**Theorem 3.8.** A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

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Also (since $T$ is linear)

$$T^{-1}(r\vec{v}_1') = T^{-1}(rT(\vec{v}_1)) = T^{-1}(T(r\vec{v}_1)) = \mathcal{I}(r\vec{v}_1) = r\vec{v}_1 = rT^{-1}(\vec{v}_1').$$

Therefore $T^{-1}$ is linear. □

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**Proof (continued).** Now we need only show that $T^{-1}$ is linear. Suppose $T(\vec{v}_1) = \vec{v}_1'$ and $T(\vec{v}_2) = \vec{v}_2'$; that is, $\vec{v}_1 = T^{-1}(\vec{v}_1')$ and $\vec{v}_2 = T^{-1}(\vec{v}_2')$. Then

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Therefore $T^{-1}$ is linear. □

# Example 3.4.C

**Example 3.4.C.** Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$, as in Example 3.4.A. Determine if $T$ is invertible. If so, find its inverse.

**Solution.** Since $\ker(T) = \{0\}$ by Example 3.4.B, then $T$ is one-to-one by Corollary 3.4.A. For any $g \in \mathcal{F}$, for $f = g/a$ we have $T(f) = T(g/a) = a(g/a) = g$ and so $T$ is onto.

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# Theorem 3.10

**Theorem 3.10. Matrix Representations of Linear Transformations.** Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$. Then the standard matrix representation of $\overline{T}$ is the matrix $A$ whose $j$th column vector is $T(\vec{b}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

**Proof.** Since $B$ is a basis for $V$ and $B$ has $n$ elements, then $\dim(V) = n$ and so by Theorem 3.3.A, "Fundamental Theorem of Finite Dimensional Vector Spaces," there is isomorphism $\alpha : V \to \mathbb{R}^n$ between $V$ and $\mathbb{R}^n$ where $\alpha(\vec{v}) = \vec{v}_B$, as shown in the proof of Theorem 3.3.A.

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We need to show for all $\vec{v} \in V$ that $T(\vec{v})_{B'} = A(\vec{v}_B)$. We are given that $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$, or equivalently

$$\overline{T}(\alpha(\vec{v})) = T(\vec{v})_{B'}. \qquad (*)$$

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# Theorem 3.10 (continued)

**Theorem 3.10. Matrix Representations of Linear Transformations.** Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$. Then the standard matrix representation of $\overline{T}$ is the matrix $A$ whose $j$th column vector is $T(\vec{b}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

**Proof (continued).** $\ldots \overline{T}(\alpha(\vec{v})) = T(\vec{v})_{B'}$. $\quad (*)$
So we need to show that $\overline{T}(\vec{v}_B) = A(\vec{v}_B)$. Since $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$, then by Corollary 2.3.A, "Standard Matrix Representation of Linear Transformations," the standard matrix representation of $\overline{T}$ is the $m \times n$ matrix whose $j$th column is $\overline{T}(\hat{e}_j)$.

# Theorem 3.10 (continued)

**Theorem 3.10. Matrix Representations of Linear Transformations.** Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$. Then the standard matrix representation of $\overline{T}$ is the matrix $A$ whose $j$th column vector is $T(\vec{b}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

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# Theorem 3.10 (continued)

**Theorem 3.10. Matrix Representations of Linear Transformations.** Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$. Then the standard matrix representation of $\overline{T}$ is the matrix $A$ whose $j$th column vector is $T(\vec{b}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

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# Page 227 Number 18

**Page 227 Number 18.** Let $V$ and $V'$ be vector spaces with ordered bases $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3, \vec{b}'_4)$, respectively. Let $T : V \to V'$ be the linear transformation having matrix representation
$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ relative to $B, B'$. Find $T(\vec{v})$ for $\vec{v} = 3\vec{b}_3 - \vec{b}_1$.

**Solution.** We use Theorem 3.10, "Matrix Representation of Linear Transformations." Notice that $\vec{v}_B = [-1, 0, 3]$.

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**Solution.** We use Theorem 3.10, "Matrix Representation of Linear Transformations." Notice that $\vec{v}_B = [-1, 0, 3]$. So

$$T(\vec{v})_{B'} = A\vec{v}_B = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 3 \\ 7 \end{bmatrix}.$$

So $\boxed{T(\vec{v}) = -7\vec{b}'_1 - 2\vec{b}'_2 + 3\vec{b}'_3 + 7\vec{b}'_4.}$ □

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So $\boxed{T(\vec{v}) = -7\vec{b}'_1 - 2\vec{b}'_2 + 3\vec{b}'_3 + 7\vec{b}'_4.}$ □

# Page 227 Number 22

**Page 227 Number 22.** Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $T(p(x)) = xD(p(x)) = xp'(x)$ and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.

**(a)** Find the matrix representation $A$ of $T$ relative to $B, B'$.

**(b)** Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

**Solution. (a)** We use Theorem 3.10, "Matrix Representation of Linear Transformations," and see that the columns of $A$ are $T(\vec{b}_1)_{B'}$, $T(\vec{b}_2)_{B'}$, $T(\vec{b}_3)_{B'}$, $T(\vec{b}_4)_{B'}$.

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**Page 227 Number 22.** Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $T(p(x)) = xD(p(x)) = xp'(x)$ and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.
**(a)** Find the matrix representation $A$ of $T$ relative to $B, B'$.
**(b)** Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

**Solution. (a)** We use Theorem 3.10, "Matrix Representation of Linear Transformations," and see that the columns of $A$ are $T(\vec{b}_1)_{B'}$, $T(\vec{b}_2)_{B'}$, $T(\vec{b}_3)_{B'}$, $T(\vec{b}_4)_{B'}$. We find

$$\begin{aligned}
T(\vec{b}_1)_{B'} &= T(x^3)_{B'} = (x(3x^2))_{B'} = (3x^3)_{B'} = [3,0,0,0]^T \\
T(\vec{b}_2)_{B'} &= T(x^2)_{B'} = (x(2x))_{B'} = (2x^2)_{B'} = [0,2,0,0]^T \\
T(\vec{b}_3)_{B'} &= T(x)_{B'} = (x(1))_{B'} = (x)_{B'} = [0,0,1,0]^T \\
T(\vec{b}_4)_{B'} &= T(1)_{B'} = (x(0))_{B'} = (0)_{B'} = [0,0,0,0]^T.
\end{aligned}$$

# Page 227 Number 22

**Page 227 Number 22.** Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $T(p(x)) = xD(p(x)) = xp'(x)$ and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.
**(a)** Find the matrix representation $A$ of $T$ relative to $B, B'$.
**(b)** Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

**Solution. (a)** We use Theorem 3.10, "Matrix Representation of Linear Transformations," and see that the columns of $A$ are $T(\vec{b}_1)_{B'}$, $T(\vec{b}_2)_{B'}$, $T(\vec{b}_3)_{B'}$, $T(\vec{b}_4)_{B'}$. We find

$$\begin{array}{ccccccccc} T(\vec{b}_1)_{B'} & = & T(x^3)_{B'} & = & (x(3x^2))_{B'} & = & (3x^3)_{B'} & = & [3,0,0,0]^T \\ T(\vec{b}_2)_{B'} & = & T(x^2)_{B'} & = & (x(2x))_{B'} & = & (2x^2)_{B'} & = & [0,2,0,0]^T \\ T(\vec{b}_3)_{B'} & = & T(x)_{B'} & = & (x(1))_{B'} & = & (x)_{B'} & = & [0,0,1,0]^T \\ T(\vec{b}_4)_{B'} & = & T(1)_{B'} & = & (x(0))_{B'} & = & (0)_{B'} & = & [0,0,0,0]^T. \end{array}$$

# Page 227 Number 22 (continued 1)

**Solution.** So $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

**(b)** First $(x^3 - 3x^2 + 4x)_{B'} = [1, -3, 4, 0]^T$. From Theorem 3.10, $T(p(x))_{B'} = A\vec{v}_B$, so we want $\vec{v}_B \in \mathbb{R}^4$ such that $A\vec{v}_B = T(p(x))_{B'} = [1, -3, 4, 0]^T$.

# Page 227 Number 22 (continued 1)

**Solution.** So $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

**(b)** First $(x^3 - 3x^2 + 4x)_{B'} = [1, -3, 4, 0]^T$. From Theorem 3.10, $T(p(x))_{B'} = A\vec{v}_B$, so we want $\vec{v}_B \in \mathbb{R}^4$ such that $A\vec{v}_B = T(p(x))_{B'} = [1, -3, 4, 0]^T$. Let $\vec{v}_B = [v_1, v_2, v_3, v_4]^T$, and consider the augmented matrix for $A\vec{v}_B = [1, -3, 4, 0]^T$: $\left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

# Page 227 Number 22 (continued 1)

**Solution.** So $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

**(b)** First $(x^3 - 3x^2 + 4x)_{B'} = [1, -3, 4, 0]^T$. From Theorem 3.10, $T(p(x))_{B'} = A\vec{v}_B$, so we want $\vec{v}_B \in \mathbb{R}^4$ such that $A\vec{v}_B = T(p(x))_{B'} = [1, -3, 4, 0]^T$. Let $\vec{v}_B = [v_1, v_2, v_3, v_4]^T$, and consider the augmented matrix for $A\vec{v}_B = [1, -3, 4, 0]^T$: $\left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

We see that this is already in row reduced echelon form and so we need

$$\begin{array}{rcl} 3v_1 & = & 1 \\ 2v_2 & = & -3 \\ v_3 & = & 4 \\ 0 & = & 0 \end{array} \quad \text{or} \quad \begin{array}{rcl} v_1 & = & 1/3 \\ v_2 & = & -3/2 \\ v_3 & = & 4 \\ v_4 & = & v_4 \end{array}.$$

# Page 227 Number 22 (continued 1)

**Solution.** So $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

**(b)** First $(x^3 - 3x^2 + 4x)_{B'} = [1, -3, 4, 0]^T$. From Theorem 3.10, $T(p(x))_{B'} = A\vec{v}_B$, so we want $\vec{v}_B \in \mathbb{R}^4$ such that $A\vec{v}_B = T(p(x))_{B'} = [1, -3, 4, 0]^T$. Let $\vec{v}_B = [v_1, v_2, v_3, v_4]^T$, and consider the augmented matrix for $A\vec{v}_B = [1, -3, 4, 0]^T$: $\left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

We see that this is already in row reduced echelon form and so we need

$$\begin{array}{rcr} 3v_1 & = & 1 \\ 2v_2 & = & -3 \\ v_3 & = & 4 \\ 0 & = & 0 \end{array} \quad \text{or} \quad \begin{array}{rcr} v_1 & = & 1/3 \\ v_2 & = & -3/2 \\ v_3 & = & 4 \\ v_4 & = & v_4 \end{array}.$$

# Page 227 Number 22 (continued 2)

**Page 227 Number 22.** Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $T(p(x)) = xD(p(x)) = xp'(x)$ and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.
**(a)** Find the matrix representation $A$ of $T$ relative to $B, B'$.
**(b)** Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

**Solution.** So we take $k = v_4$ as a free variable. Then $\vec{v}_B = [1/3, -3/2, 4, k]$ for any $k \in \mathbb{R}$. So $\vec{v} \in \mathcal{P}_3$ is of the form

$$\boxed{\frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x + k \text{ for } k \in \mathbb{R}.} \quad \square$$

# Page 227 Number 22 (continued 2)

**Page 227 Number 22.** Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $T(p(x)) = xD(p(x)) = xp'(x)$ and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.
**(a)** Find the matrix representation $A$ of $T$ relative to $B, B'$.
**(b)** Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

**Solution.** So we take $k = v_4$ as a free variable. Then $\vec{v}_B = [1/3, -3/2, 4, k]$ for any $k \in \mathbb{R}$. So $\vec{v} \in \mathcal{P}_3$ is of the form

$$\boxed{\frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x + k \text{ for } k \in \mathbb{R}.} \quad \square$$

# Page 227 Number 24

**Page 227 Number 24.** Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ be defined by $T(p(x)) = p'(x)|_{2x+1} = p'(2x+1)$, where $p'(x) = D(p(x))$, and let $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4) = (x^3, x^2, x, 1)$ and $B' = (x^2, x, 1) = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3)$.

**(a)** Find the matrix representation $A$ of $T$ relative to $B, B'$.

**(b)** Use $A$ from part (a) to compute $T(4x^3 - 5x^2 + 4x - 7)$.

**Solution. (a)** Again we use Theorem 3.10 and find $T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}, T(\vec{b}_4)_{B'}$. First we need the derivatives of $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4$: $\frac{d}{dx}[\vec{b}_1] = \frac{d}{dx}[x^3] = 3x^2$, $\frac{d}{dx}[\vec{b}_2] = \frac{d}{dx}[x^2] = 2x$, $\frac{d}{dx}[\vec{b}_3] = \frac{d}{dx}[x] = 1$, and $\frac{d}{dx}[\vec{b}_4] = \frac{d}{dx}[1] = 0$.

# Page 227 Number 24

**Page 227 Number 24.** Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ be defined by $T(p(x)) = p'(x)|_{2x+1} = p'(2x+1)$, where $p'(x) = D(p(x))$, and let $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4) = (x^3, x^2, x, 1)$ and $B' = (x^2, x, 1) = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3)$.
**(a)** Find the matrix representation $A$ of $T$ relative to $B, B'$.
**(b)** Use $A$ from part (a) to compute $T(4x^3 - 5x^2 + 4x - 7)$.

**Solution. (a)** Again we use Theorem 3.10 and find $T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}, T(\vec{b}_4)_{B'}$. First we need the derivatives of $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4$: $\frac{d}{dx}[\vec{b}_1] = \frac{d}{dx}[x^3] = 3x^2$, $\frac{d}{dx}[\vec{b}_2] = \frac{d}{dx}[x^2] = 2x$, $\frac{d}{dx}[\vec{b}_3] = \frac{d}{dx}[x] = 1$, and $\frac{d}{dx}[\vec{b}_4] = \frac{d}{dx}[1] = 0$. Since $T$ first takes a derivative and then evaluates it at $2x+1$, we have $T(x^3) = 3(2x+1)^2 = 12x^2 + 12x + 3$, $T(x^2) = 2(2x+1) = 4x+2$, $T(x) = 1$, and $T(1) = 0$, and so
$T(\vec{b}_1)_{B'} = T(x^3)_{B'} = (12x^2 + 12x + 3)_{B'} = [12, 12, 3]^T$,
$T(\vec{b}_2)_{B'} = T(x^2)_{B'} = (4x+2)_{B'} = [0, 4, 2]^T$,
$T(\vec{b}_3)_{B'} = T(x)_{B'} = (1)_{B'} = [0, 0, 1]^T$, and
$T(\vec{b}_4)_{B'} = T(1)_{B'} = 0_{B'} = [0, 0, 0]^T$.

# Page 227 Number 24

**Page 227 Number 24.** Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ be defined by $T(p(x)) = p'(x)|_{2x+1} = p'(2x+1)$, where $p'(x) = D(p(x))$, and let $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4) = (x^3, x^2, x, 1)$ and $B' = (x^2, x, 1) = (\vec{b}_1', \vec{b}_2', \vec{b}_3')$.
**(a)** Find the matrix representation $A$ of $T$ relative to $B, B'$.
**(b)** Use $A$ from part (a) to compute $T(4x^3 - 5x^2 + 4x - 7)$.

**Solution. (a)** Again we use Theorem 3.10 and find $T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}, T(\vec{b}_4)_{B'}$. First we need the derivatives of $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4$: $\frac{d}{dx}[\vec{b}_1] = \frac{d}{dx}[x^3] = 3x^2$, $\frac{d}{dx}[\vec{b}_2] = \frac{d}{dx}[x^2] = 2x$, $\frac{d}{dx}[\vec{b}_3] = \frac{d}{dx}[x] = 1$, and $\frac{d}{dx}[\vec{b}_4] = \frac{d}{dx}[1] = 0$. Since $T$ first takes a derivative and then evaluates it at $2x+1$, we have
$T(x^3) = 3(2x+1)^2 = 12x^2 + 12x + 3$, $T(x^2) = 2(2x+1) = 4x + 2$, $T(x) = 1$, and $T(1) = 0$, and so
$T(\vec{b}_1)_{B'} = T(x^3)_{B'} = (12x^2 + 12x + 3)_{B'} = [12, 12, 3]^T$,
$T(\vec{b}_2)_{B'} = T(x^2)_{B'} = (4x + 2)_{B'} = [0, 4, 2]^T$,
$T(\vec{b}_3)_{B'} = T(x)_{B'} = (1)_{B'} = [0, 0, 1]^T$, and
$T(\vec{b}_4)_{B'} = T(1)_{B'} = 0_{B'} = [0, 0, 0]^T$.

# Page 227 Number 24 (continued 1)

**Solution.** So the columns of $A$ are $T(\vec{b}_1)$, $T(\vec{b}_2)$, $T(\vec{b}_3)$, $T(\vec{b}_4)$:
$A = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$. □

**(b)** We know from Theorem 3.10, "Matrix Representations of Linear Transformations," that $T(4x^3 - 5x^2 + 4x - 7)_{B'} = A\vec{v}_B$. Now $\vec{v}_B = [4, -5, 4, -7]$ so

$$T(4x^3 - 5x^2 + 4x - 7)_{B'} = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 48 \\ 28 \\ 6 \end{bmatrix}$$

and hence

$$\boxed{T(4x^3 - 5x^2 + 4x - 7) = (48)x^2 + (28)x + (6)1 = 48x^2 + 28x + 6.}$$

# Page 227 Number 24 (continued 1)

**Solution.** So the columns of $A$ are $T(\vec{b}_1)$, $T(\vec{b}_2)$, $T(\vec{b}_3)$, $T(\vec{b}_4)$:

$$A = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}. \square$$

**(b)** We know from Theorem 3.10, "Matrix Representations of Linear Transformations," that $T(4x^3 - 5x^2 + 4x - 7)_{B'} = A\vec{v}_B$. Now $\vec{v}_B = [4, -5, 4, -7]$ so

$$T(4x^3 - 5x^2 + 4x - 7)_{B'} = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 48 \\ 28 \\ 6 \end{bmatrix}$$

and hence

$$\boxed{T(4x^3 - 5x^2 + 4x - 7) = (48)x^2 + (28)x + (6)1 = 48x^2 + 28x + 6.}$$

# Page 227 Number 24 (continued 2)

**Page 227 Number 24.** Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ be defined by $T(p(x)) = p'(x)|_{2x+1} = p'(2x+1)$, where $p'(x) = D(p(x))$, and let $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4) = (x^3 + x^2, x, 1)$ and $B' = (x^2, x, 1) = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3)$.
**(b)** Use $A$ from part (a) to compute $T(4x^3 - 5x^2 + 4x - 7)$.

**Solution.** Notice that $\frac{d}{dx}[4x^3 - 5x^2 + 4x - 7] = 12x^2 - 10x + 4$ and evaluating this at $2x + 1$ gives

$$12(2x+1)^2 - 10(2x+1) + 4 = 12(4x^2 + 4x + 1) - 10(2x+1) + 4$$

$$= 48x^2 + 48x + 12 - 20x - 10 + 4 = 48x^2 + 28x + 6,$$

as expected. □

# Page 228 Number 28

**Page 228 Number 28.** Let $W = \text{sp}(e^{2x}, e^{4x}, e^{8x})$ be a subspace of $\mathcal{F}$ (see Example 3.1.3) and let $B = B' = (e^{2x}, e^{4x}, e^{8x})$.
**(a)** Find the matrix representation $A$ relative to $B, B'$ of the linear transformation $T : W \to W$ defined by $T(f) = \int_{-\infty}^{x} f(t)\,dt$.
**(b)** Find $A^{-1}$ where $A$ is the matrix of part (a) and use it to find $T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})$.

**Solution. (a)** We use Theorem 3.10 and find $T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}$. We have

$$T(\vec{b}_1) = T(e^{2x}) = \int_{-\infty}^{x} e^{2t}\,dt = \lim_{a\to-\infty}\left(\int_a^x e^{2t}\,dt\right) = \lim_{a\to-\infty}\left(\left(\frac{1}{2}e^{2t}\right)\Big|_a^x\right)$$

$$= \lim_{a\to-\infty}\left(\frac{1}{2}e^{2x} - \frac{1}{2}e^{2a}\right) = \frac{1}{2}e^{2x} - 0 = \frac{1}{2}e^{2x}$$

# Page 228 Number 28

**Page 228 Number 28.** Let $W = \mathrm{sp}(e^{2x}, e^{4x}, e^{8x})$ be a subspace of $\mathcal{F}$ (see Example 3.1.3) and let $B = B' = (e^{2x}, e^{4x}, e^{8x})$.
**(a)** Find the matrix representation $A$ relative to $B, B'$ of the linear transformation $T : W \to W$ defined by $T(f) = \int_{-\infty}^{x} f(t)\, dt$.
**(b)** Find $A^{-1}$ where $A$ is the matrix of part (a) and use it to find $T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})$.

**Solution.** **(a)** We use Theorem 3.10 and find $T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}$. We have

$$T(\vec{b}_1) = T(e^{2x}) = \int_{-\infty}^{x} e^{2t}\, dt = \lim_{a \to -\infty} \left( \int_{a}^{x} e^{2t}\, dt \right) = \lim_{a \to -\infty} \left( \left( \frac{1}{2} e^{2t} \right) \Big|_{a}^{x} \right)$$

$$= \lim_{a \to -\infty} \left( \frac{1}{2} e^{2x} - \frac{1}{2} e^{2a} \right) = \frac{1}{2} e^{2x} - 0 = \frac{1}{2} e^{2x}$$

# Page 228 Number 28 (continued 1)

**Solution (continued).**

$$T(\vec{b}_2) = T(e^{4x}) = \int_{-\infty}^{x} e^{4t}\,dt = \lim_{a\to-\infty}\left(\int_a^x e^{4t}\,dt\right) = \lim_{a\to-\infty}\left(\left(\frac{1}{4}e^{4t}\right)\Big|_a^x\right)$$

$$= \lim_{a\to-\infty}\left(\frac{1}{4}e^{4x} - \frac{1}{4}e^{4a}\right) = \frac{1}{4}e^{4x} - 0 = \frac{1}{4}e^{4x}$$

$$T(\vec{b}_3) = T(e^{8x}) = \int_{-\infty}^{x} e^{8t}\,dt = \lim_{a\to-\infty}\left(\int_a^x e^{8t}\,dt\right) = \lim_{a\to-\infty}\left(\left(\frac{1}{8}e^{t}\right)\Big|_a^x\right)$$

$$= \lim_{a\to-\infty}\left(\frac{1}{8}e^{8x} - \frac{1}{8}e^{8a}\right) = \frac{1}{8}e^{8x} - 0 = \frac{1}{8}e^{8x}.$$

So $T(\vec{b}_1)_{B'} = [1/2, 0, 0]$, $T(\vec{b}_2)_{B'} = [0, 1/4, 0]$, $T(\vec{b}_3)_{B'} = [0, 0, 1/8]$. So

$$\boxed{A = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}}.$$

# Page 228 Number 28 (continued 2)

**Page 228 Number 28.** Let $W = \text{sp}(e^{2x}, e^{4x}, e^{8x})$ a subspace of $\mathcal{F}$ (see Example 3.1.3) and let $B = B' = (e^{2x}, e^{4x}, e^{8x})$.
**(b)** Find $A^{-1}$ where $A$ is the matrix of part (a) and use it to find $T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})$.

**Solution (continued). (b)** It is easy to see that $A^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$. By Theorem 3.4.B, $A^{-1}$ is the matrix representation of $T^{-1}$ relative to $B', B$. So by Theorem 3.10, "Matrix Representations of Linear Transformations," we have that $T^{-1}(\vec{v})_B = A^{-1}\vec{v}_{B'}$ and so

$$T^{-1}(r_1 e^{2x}+r_2 e^{4x}+r_3 e^{8x})_B = A^{-1}((r_1 e^{2x}+r_2 e^{4x}+r_3 e^{8x})'_B) = A^{-1}[r_1, r_2, r_3]^T$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2r_1 \\ 4r_2 \\ 8r_3 \end{bmatrix}.$$

# Page 228 Number 28 (continued 2)

**Page 228 Number 28.** Let $W = \mathrm{sp}(e^{2x}, e^{4x}, e^{8x})$ a subspace of $\mathcal{F}$ (see Example 3.1.3) and let $B = B' = (e^{2x}, e^{4x}, e^{8x})$.
**(b)** Find $A^{-1}$ where $A$ is the matrix of part (a) and use it to find $T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})$.

**Solution (continued). (b)** It is easy to see that $A^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$. By Theorem 3.4.B, $A^{-1}$ is the matrix representation of $T^{-1}$ relative to $B', B$. So by Theorem 3.10, "Matrix Representations of Linear Transformations," we have that $T^{-1}(\vec{v})_B = A^{-1}\vec{v}_{B'}$ and so

$$T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})_B = A^{-1}((r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})'_B) = A^{-1}[r_1, r_2, r_3]^T$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2r_1 \\ 4r_2 \\ 8r_3 \end{bmatrix}.$$

# Page 228 Number 28 (continued 3)

**Page 228 Number 28.** Let $W = \text{sp}(e^{2x}, e^{4x}, e^{8x})$ a subspace of $\mathcal{F}$ (see Example 3.1.3) and let $B = B' = (e^{2x}, e^{4x}, e^{8x})$.
**(b)** Find $A^{-1}$ where $A$ is the matrix of part (a) and use it to find $T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})$.

**Solution (continued).** . . .

$$T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})_B = \begin{bmatrix} 2r_1 \\ 4r_2 \\ 8r_3 \end{bmatrix}.$$

So translating this using basis $B$ we have

$$\boxed{T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x}) = 2r_1 e^{2x} + 4r_2 e^{4x} + 8r_3 e^{8x}.} \square$$

# Page 229 Number 44

**Page 229 Number 44.** Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

**Solution.** Let $\vec{v}_1, \vec{v}_2 \in V$ and $s, t \in \mathbb{R}$ be scalars.

# Page 229 Number 44

**Page 229 Number 44.** Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

**Solution.** Let $\vec{v}_1, \vec{v}_2 \in V$ and $s, t \in \mathbb{R}$ be scalars. Then

$$\begin{aligned}
(rT)(s\vec{v}_1 + t\vec{v}_2) &= r(T(s\vec{v}_1 + t\vec{v}_2)) \text{ by the definition of } rT \\
&= r(sT(\vec{v}_1) + tT(\vec{v}_2)) \text{ by Note 3.4.A since } T \text{ is linear} \\
&= r(sT(\vec{v}_1)) + r(tT(\vec{v}_2)) \text{ by S1} \\
&= (rs)T(\vec{v}_1) + (rt)T(\vec{v}_2) \text{ by S3}
\end{aligned}$$

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&= r(sT(\vec{v}_1)) + r(tT(\vec{v}_2) \text{ by S1} \\
&= (rs)T(\vec{v}_1) + (rt)T(\vec{v}_2) \text{ by S3} \\
&= (sr)T(\vec{v}_1) + (tr)T(\vec{v}_2) \text{ since multiplication} \\
&\quad \text{ is commutative in } \mathbb{R} \\
&= s(rT(\vec{v}_1)) + t(rT(\vec{v}_2)) \text{ by S3} \\
&= s(rT)(\vec{v}_1) + t(rT)(\vec{v}_2) \text{ by definition of } rT.
\end{aligned}$$

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\end{aligned}$$

# Page 229 Number 44 (continued)

**Page 229 Number 44.** Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

**Solution (continued).** So $rT$ is a linear transformation by Note 3.4.A. □

# Page 229 Number 44 (continued)

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**Solution (continued).** So $rT$ is a linear transformation by Note 3.4.A. $\square$

**Note.** In Exercise 43 it is shown for $T_1, T_2 \in L(V, V')$ that $T_1 + T_2 \in L(V, V')$ where we define $(T_1 + T_2)(\vec{v}_1 + \vec{v}_2) = T_1(\vec{v}_1) + T_2(\vec{v}_2)$. So $L(V, V')$ is closed under vector addition and scalar multiplication.

# Page 229 Number 44 (continued)

**Page 229 Number 44.** Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

**Solution (continued).** So $rT$ is a linear transformation by Note 3.4.A. □

**Note.** In Exercise 43 it is shown for $T_1, T_2 \in L(V, V')$ that $T_1 + T_2 \in L(V, V')$ where we define $(T_1 + T_2)(\vec{v}_1 + \vec{v}_2) = T_1(\vec{v}_1) + T_2(\vec{v}_2)$. So $L(V, V')$ is closed under vector addition and scalar multiplication. Therefore, by Theorem 3.2, "Test for a Subspace," $L(V, V')$ is a subspace of the vector space of all functions mapping $V$ into $V'$ (see "Summary Item 5 on page 188).

# Page 229 Number 44 (continued)

**Page 229 Number 44.** Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

**Solution (continued).** So $rT$ is a linear transformation by Note 3.4.A. □

**Note.** In Exercise 43 it is shown for $T_1, T_2 \in L(V, V')$ that $T_1 + T_2 \in L(V, V')$ where we define $(T_1 + T_2)(\vec{v}_1 + \vec{v}_2) = T_1(\vec{v}_1) + T_2(\vec{v}_2)$. So $L(V, V')$ is closed under vector addition and scalar multiplication. Therefore, by Theorem 3.2, "Test for a Subspace," $L(V, V')$ is a subspace of the vector space of all functions mapping $V$ into $V'$ (see "Summary Item 5 on page 188).

# Page 226 Number 12

**Page 226 Number 12.** Let $D_\infty$ be the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}$ that have derivatives of all orders. It can be shown that the kernel of a linear transformation $T : D_\infty \to D_\infty$ of the form $T(f) = a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f$, where $a_n \neq 0$, is an $n$-dimensional subspace of $D_\infty$. Use this information to find the solution set in $D_\infty$ of the differential equation $y' - y = x$. HINT: a particular solution to the differential equation is $y = -x - 1$.

**Solution.** First, we consider the "homogeneous" linear differential equation $y' - y = 0$; that is, $y' = y$. We know from Calculus that if $y' = y$ then $y = ke^x$ for some $k \in \mathbb{R}$ ($y' = y$ is a separable differential equation and can be solved by separation of variables and integration). This is the general solution to $y' - y = 0$ and the set of all such solutions form a subspace of the vector space $\mathcal{F}$ of all real valued functions defined on $\mathbb{R}$ (see exercise 3.2.40).

# Page 226 Number 12

**Page 226 Number 12.** Let $D_\infty$ be the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}$ that have derivatives of all orders. It can be shown that the kernel of a linear transformation $T : D_\infty \to D_\infty$ of the form $T(f) = a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f$, where $a_n \neq 0$, is an $n$-dimensional subspace of $D_\infty$. Use this information to find the solution set in $D_\infty$ of the differential equation $y' - y = x$. HINT: a particular solution to the differential equation is $y = -x - 1$.

**Solution.** First, we consider the "homogeneous" linear differential equation $y' - y = 0$; that is, $y' = y$. We know from Calculus that if $y' = y$ then $y = ke^x$ for some $k \in \mathbb{R}$ ($y' = y$ is a separable differential equation and can be solved by separation of variables and integration). This is the general solution to $y' - y = 0$ and the set of all such solutions form a subspace of the vector space $\mathcal{F}$ of all real valued functions defined on $\mathbb{R}$ (see exercise 3.2.40).

# Page 226 Number 12 (continued)

**Page 226 Number 12.** Let $D_\infty$ be the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}$ that have derivatives of all orders. It can be shown that the kernel of a linear transformation $T : D_\infty \to D_\infty$ of the form $T(f) = a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f$, where $a_n \neq 0$, us an $n$-dimensional subspace of $D_\infty$. Use this information to find the solution set in $D_\infty$ of the differential equation $y' - y = x$. HINT: a particular solution to the differential equation is $y = -x - 1$.

**Solution (continued).** By the solution to Exercise 3.2.41, all solutions to $y' - y = x$ are of the form $p(x) + h(x)$ where $p(x)$ is a particular solution to $y' - y = x$ and $h(x)$ is some solution to the homogeneous differential equation $y' - y = 0$. We are given that a particular solution to $y' - y = x$ is $y = -x - 1$. So the solution set to the differential equation $y' - y = x$ is $\boxed{\{-x - 1 + ke^x \mid k \in \mathbb{R}\}.}$ □