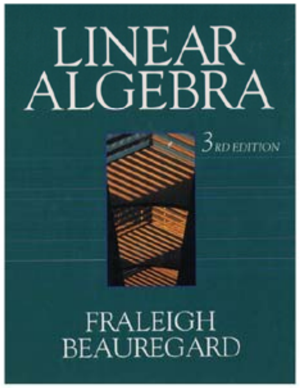
Linear Algebra

Chapter 3. Vector Spaces

Section 3.5. Inner-Product Spaces—Proofs of Theorems



Page 236 Number 2

Page 236 Number 2. For $[x_1, x_2], [y_1, y_2] \in \mathbb{R}^2$, define the quantity $\langle [x_1, x_2], [y_1, y_2] \rangle = x_1 x_2 + y_1 y_2$. Does this satisfy the conditions for an inner product on $\mathbb{R}^2$?

Solution. We test the parts of Definition 3.12, "Inner-Product Space."

P1.

$$\langle [y_1, y_2], [x_1, x_2] \rangle = y_1 y_2 + x_1 x_2 = x_1 x_2 + y_1 y_2 = \langle [x_1, x_2], [y_1, y_2] \rangle.$$

So P1 is satisfied.

P2. Let $[z_1, z_2] \in \mathbb{R}^2$. Then

$$\langle [z_1, z_2], [x_1, x_2] + [y_1, y_2] \rangle = \langle [z_1, z_2], [x_1 + y_1, x_2 + y_2] \rangle$$

$$= z_1 z_2 + (x_1 + y_1)(x_2 + y_2) = z_1 z_2 + (x_1 x_2 + y_1 x_2 + x_1 y_2 + y_1 y_2)$$

and

$$\langle [z_1, z_2], [x_1, x_2] \rangle + \langle [z_1, z_2], [y_1, y_2] \rangle = (z_1 z_2 + x_1 x_2) + (z_1 z_2 + y_1 y_2).$$

So these don't appear to be the same.

Page 236 Number 2 (continued)

Page 236 Number 2. For $[x_1, x_2], [y_1, y_2] \in \mathbb{R}^2$, define the quantity $\langle [x_1, x_2], [y_1, y_2] \rangle = x_1 x_2 + y_1 y_2$. Does this satisfy the conditions for an inner product on $\mathbb{R}^2$?

Solution (continued). We pick particular numbers (inspired by these equations) to show that P2 does not hold. Consider $[z_1, z_2] = [0, 0]$, $[x_1, x_2] = [1, 1]$, and $[y_1, y_2] = [2, 2]$. Then

$$\langle [0, 0], [1, 1] + [2, 2] \rangle = \langle [0, 0], [3, 3] \rangle = (0)(0) + (3)(3) = 9$$

and

$$\langle [0, 0], [1, 1] \rangle + \langle [0, 0], [2, 2] \rangle = ((0)(0) + (1)(1)) + ((0)(0) + (2)(2))$$

$$= 1 + 4 = 5 \neq 9 = \langle [0, 0], [1, 1] + [2, 2] \rangle.$$

So P2 does not hold and this is $\boxed{\text{not an inner product in } \mathbb{R}^2.}$ □

Page 236 Number 10

Page 236 Number 10. Let $C_{a,b}$ be the vector space of all continuous real valued functions with domain $a \le x \le b$ (see Note 3.1.A for a related vector space). Define $\langle f, g \rangle = \int_a^b f(z)g(x)\,dx$. Prove that $\langle \cdot, \cdot \rangle$ is an inner product on $C_{a,b}$.

Proof. We show that $\langle \cdot, \cdot \rangle$ satisfies P1–P4 of Definition 3.12, "Inner-Product Space." Let $f, g, h \in C_{a,b}$ and $r \in \mathbb{R}$. Then

P1. $\langle f, g \rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g, f \rangle$.

P2. $\langle f, g + h \rangle = \int_a^b f(x)(g(x) + h(x))\,dx = \int_a^b (f(x)g(x) + f(x)h(x))\,dx$

$$= \int_a^b f(x)g(x)\,dx + \int_a^b f(x)h(x)\,dx = \langle f, g \rangle + \langle f, h \rangle.$$

P3. $r\langle f, g \rangle = r\int_a^b f(x)g(x)\,dx = \int_a^b rf(x)g(x)\,dx$

$$= \begin{cases} \int_a^b (rf(x))g(x)\,dx = \langle rf, g \rangle \\ \int_a^b f(x)(rg(x))\,dx = \langle f, rg \rangle. \end{cases}$$

Page 236 Number 10 (continued)

Page 236 Number 10. Let $C_{a,b}$ be the vector space of all continuous real valued functions with domain $a \le x \le b$ (see Note 3.1.A for a related vector space). Define $\langle f, g\rangle = \int_a^b f(z)g(x)\,dx$. Prove that $\langle \cdot, \cdot \rangle$ is an inner product on $C_{a,b}$.

Proof (continued).
P4. $\langle f, f\rangle = \int_a^b f(x)f(x)\,dx = \int_a^b (f(x))^2\,dx \ge 0$, and $\langle f, f\rangle = \int_a^b (f(x))^2\,dx = 0$ if and only if $(f(x))^2 = 0$ for all $a \le x \le b$, that is if and only if $f(x) = 0$ for $a \le x \le b$ (in $C_{a,b}$ this means that f is the zero vector).

So $\langle \cdot, \cdot \rangle$ satisfies P1-P4 and hence is an inner product on $C_{a,b}$. □

Theorem 3.11

Theorem 3.11. Schwarz Inequality.
Let V be an inner-product space, and let $\vec{v}, \vec{w} \in V$. Then $\langle \vec{v}, \vec{w}\rangle \le \|\vec{v}\|\|\vec{w}\|$.

Proof. Let $r, s \in \mathbb{R}$. Then we have:

$$\begin{aligned}
\|r\vec{v} + s\vec{w}\|^2 &= \langle r\vec{v} + s\vec{w}, r\vec{v} + s\vec{w}\rangle \text{ by Definition 3.13, ``norm''}\\
&= \langle r\vec{v} + s\vec{w}, r\vec{v}\rangle + \langle r\vec{v} + s\vec{w}, s\vec{w}\rangle \text{ by P2}\\
&= \langle r\vec{v}, r\vec{v} + s\vec{w}\rangle + \langle s\vec{w}, r\vec{v} + s\vec{w}\rangle \text{ by P1}\\
&= \langle r\vec{v}, r\vec{v}\rangle + \langle r\vec{v}, s\vec{w}\rangle + \langle s\vec{w}, r\vec{v}\rangle + \langle s\vec{w}, s\vec{w}\rangle \text{ by P2}\\
&= r^2\langle \vec{v}, \vec{v}\rangle + 2rs\langle \vec{v}, \vec{w}\rangle + s^2\langle \vec{w}, \vec{w}\rangle \text{ by P1 and P3}\\
&\ge 0 \text{ by P4.}
\end{aligned}$$

Since this equation holds for all $r, s \in \mathbb{R}$, we are free to choose particular values of r and s. We choose $r = \langle \vec{w}, \vec{w}\rangle$ and $s = -\langle \vec{v}, \vec{w}\rangle$.

Theorem 3.11 (continued)

Proof (continued). ...

$$r^2\langle \vec{v}, \vec{v}\rangle + 2rs\langle \vec{v}, \vec{w}\rangle + s^2\langle \vec{w}, \vec{w}\rangle \ge 0$$

with $r = \langle \vec{w}, \vec{w}\rangle$ and $s = -\langle \vec{v}, \vec{w}\rangle$ we have

$$\langle \vec{w}, \vec{w}\rangle^2\langle \vec{v}, \vec{v}\rangle - 2\langle \vec{w}, \vec{w}\rangle\langle \vec{v}, \vec{w}\rangle^2 + \langle \vec{v}, \vec{w}\rangle^2\langle \vec{w}, \vec{w}\rangle$$

$$= \langle \vec{w}, \vec{w}\rangle^2\langle \vec{v}, \vec{v}\rangle - \langle \vec{w}, \vec{w}\rangle\langle \vec{v}, \vec{w}\rangle^2 = \langle \vec{w}, \vec{w}\rangle[\langle \vec{w}, \vec{w}\rangle\langle \vec{v}, \vec{v}\rangle - \langle \vec{v}, \vec{w}\rangle^2] \ge 0. \quad (13)$$

If $\langle \vec{w}, \vec{w}\rangle = 0$ then $\vec{w} = \vec{0}$ by Theorem 3.12 Part (P4), and the Schwarz Inequality is proven (since it reduces to $0 \ge 0$). If $\|\vec{w}\|^2 = \langle \vec{w}, \vec{w}\rangle \ne 0$, then by the above inequality the other factor of inequality (13) must also be nonnegative:

$$\langle \vec{w}, \vec{w}\rangle\langle \vec{v}, \vec{v}\rangle - \langle \vec{v}, \vec{w}\rangle^2 \ge 0.$$

Therefore

$$\langle \vec{v}, \vec{w}\rangle^2 \le \langle \vec{v}, \vec{v}\rangle\langle \vec{w}, \vec{w}\rangle = \|\vec{v}\|^2\|\vec{w}\|^2.$$

Taking square roots, we get the Schwarz Inequality. □

Page 236 Number 12

Page 236 Number 12. Show that $\sin x$ and $\cos x$ are orthogonal functions in the vector space $C_{0,\pi}$ of continuous functions with domain $0 \le x \le \pi$ where the inner product is defined as $\langle f, g\rangle = \int_0^\pi f(x)g(x)\,dx$ (see Exercise 10).

Solution. We have (by u-substitution with $u = \sin x$ and $du = \cos x$):

$$\langle \cos x, \sin x\rangle = \int_0^\pi \cos x \sin x\,dx$$

$$= \frac{1}{2}\sin^2 x\Big|_0^\pi = \frac{1}{2}\sin^2\pi - \frac{1}{2}\sin^2 0 = 0 - 0 = 0.$$

So by the definition of orthogonal in an inner-product space, $\cos x$ and $\sin x$ are orthogonal. □

Page 237 Number 18

Page 237 Number 18. For vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in an inner-product space and for scalars r and s, prove that if $\vec{w}$ is perpendicular to both $\vec{u}$ and $\vec{v}$ then $\vec{w}$ is perpendicular to $r\vec{u} + s\vec{v}$.

Proof. Since $\vec{w}$ is perpendicular to both $\vec{u}$ and $\vec{v}$ then by the definition of perpendicular (or orthogonal), $\langle \vec{w}, \vec{u} \rangle = \langle \vec{w}, \vec{v} \rangle = 0$. So

$$\begin{aligned} \langle \vec{w}, r\vec{u} + s\vec{v} \rangle &= \langle \vec{w}, r\vec{u} \rangle + \langle \vec{w}, s\vec{v} \rangle \text{ by P2} \\ &= r\langle \vec{w}, \vec{u} \rangle + s\langle \vec{w}, \vec{v} \rangle \text{ by P3} \\ &= r(0) + s(0) = 0. \end{aligned}$$

So $\vec{w}$ is perpendicular to $r\vec{u} + s\vec{v}$, as claimed. □

Page 237 Number 20

Page 237 Number 20. Let V be an inner-product space and let S be a subset of V. Prove that

$$S^\perp = \{\vec{v} \in V \mid \vec{v} \text{ is orthogonal to each vector in } S\}$$

is a subspace of V. $S^\perp$ is called the *perp space* of set S.

Proof. We apply Theorem 3.2, "Test for a Subspace," and test $S^\perp$ for closure under vector addition and closure under scalar multiplication. Let $\vec{v}, \vec{w} \in S^\perp$ and let $r \in \mathbb{R}$ be a scalar. Then by the definition of $S^\perp$, for every $\vec{s} \in S$ we have $\langle \vec{v}, \vec{s} \rangle = 0$ and $\langle \vec{w}, \vec{s} \rangle = 0$. Now

$$\begin{aligned} \langle \vec{v} + \vec{w}, \vec{s} \rangle &= \langle \vec{s}, \vec{v} + \vec{w} \rangle \text{ by P1} \\ &= \langle \vec{s}, \vec{v} \rangle + \langle \vec{s}, \vec{w} \rangle \text{ by P2} \\ &= 0 + 0 = 0 \end{aligned}$$

for every $\vec{s} \in S$ and hence $\vec{v} + \vec{w} \in S^\perp$ and $S^\perp$ is closed under vector addition.

Page 237 Number 20 (continued)

Page 237 Number 20. Let V be an inner-product space and let S be a subset of V. Prove that

$$S^\perp = \{\vec{v} \in V \mid \vec{v} \text{ is orthogonal to each vector in } S\}$$

is a subspace of V. $S^\perp$ is called the *perp space* of set S.

Proof (continued). Next,

$$\begin{aligned} \langle r\vec{v}, \vec{s} \rangle &= r\langle \vec{v}, \vec{s} \rangle \text{ by P3} \\ &= r(0) = 0 \end{aligned}$$

for every $\vec{s} \in S$ and hence $r\vec{v} \in S^\perp$ and $S^\perp$ is closed under scalar multiplication. Therefore Theorem 3.2, "Test for Subspace," implies that $S^\perp$ is a subspace of V. □

Page 237 Number 24

Page 237 Number 24. Use the Triangle Inequality to prove that for any $\vec{v}, \vec{w}$ in an inner-product space V, $\|\vec{v} - \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$.

Proof. Let $\vec{v}, \vec{w} \in V$. Then $-\vec{w} \in V$ and so we consider the vector sum $\vec{v} + (-\vec{w})$. The Triangle Inequality implies that $\|\vec{v} + (-\vec{w})\| \le \|\vec{v}\| + \|-\vec{w}\|$. Now

$$\begin{aligned} \|-\vec{w}\| &= \sqrt{\langle -\vec{w}, -\vec{w} \rangle} \text{ by Definition 3.13,} \\ &\quad \text{Magnitude or Norm of a Vector''} \\ &= \sqrt{(-1)\langle \vec{w}, -\vec{w} \rangle} = \sqrt{(-1)(-1)\langle \vec{w}, \vec{w} \rangle} \text{ by P3} \\ &= \sqrt{\langle \vec{w}, \vec{w} \rangle} = \|\vec{w}\|. \end{aligned}$$

So $\|\vec{v} - \vec{w}\| = \|\vec{v} + (-\vec{w})\| \le \|\vec{v}\| + \|-\vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$, as claimed. □

Page 237 Number 26

Page 237 Number 26. Let $\vec{v}$ and $\vec{w}$ be vectors in an inner-product space V. Show that $\|\vec{v}\|\vec{w} + \|\vec{w}\|\vec{v}$ is perpendicular to $\|\vec{v}\|\vec{w} - \|\vec{w}\|\vec{v}$.

Solution. Consider

$$
\begin{aligned}
&\langle \|\vec{v}\|\vec{w} + \|\vec{w}\|\vec{v}, \|\vec{v}\|\vec{w} - \|\vec{w}\|\vec{v}\rangle \\
= \ & \langle \|\vec{v}\|\vec{w} + \|\vec{w}\|\vec{v}, \|\vec{v}\|\vec{w}\rangle + \langle \|\vec{v}\|\vec{w} + \|\vec{w}\|\vec{v}, -\|\vec{w}\|\vec{v}\rangle \text{ by P2} \\
= \ & \langle \|\vec{v}\|\vec{w}, \|\vec{v}\|\vec{w} + \|\vec{w}\|\vec{v}\rangle + \langle -\|\vec{w}\|\vec{v}, \|\vec{v}\|\vec{w} + \|\vec{w}\|\vec{v}\rangle \text{ by P1} \\
= \ & \langle \|\vec{v}\|\vec{w}, \|\vec{v}\|\vec{w}\rangle + \langle \|\vec{v}\|\vec{w}, \|\vec{w}\|\vec{v}\rangle + \langle -\|\vec{w}\|\vec{v}, \|\vec{v}\|\vec{w}\rangle \\
& + \langle -\|\vec{w}\|\vec{v}, \|\vec{w}\|\vec{v}\rangle \text{ by P2} \\
= \ & \|\vec{v}\|^2\langle \vec{w}, \vec{w}\rangle + \|\vec{v}\|\|\vec{w}\|\langle \vec{w}, \vec{v}\rangle - \|\vec{v}\|\|\vec{w}\|\langle \vec{v}, \vec{w}\rangle - \|\vec{w}\|^2\langle \vec{v}, \vec{v}\rangle \text{ by P3} \\
= \ & \|\vec{v}\|^2\|\vec{w}\|^2 - \|\vec{w}\|^2\|\vec{v}\|^2 \text{ by Definition 3.13,} \\
& \text{"Magnitude and Norm of a Vector"} \\
= \ & 0.
\end{aligned}
$$

Since the inner product is 0, the vectors are perpendicular. □