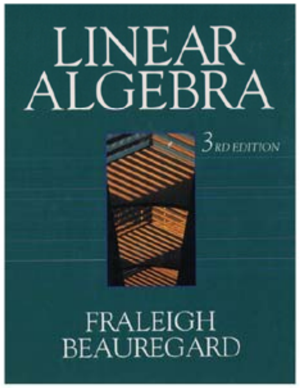
Linear Algebra

Chapter 4: Determinants

Section 4.1. Areas, Volumes, and Cross Products—Proofs of Theorems



Page 248 Number 20

Page 248 Number 20. Find the area of the parallelogram with vertex at the origin and with vectors $-\hat{\imath} + 4\hat{\jmath}$ and $2\hat{\imath} + 3\hat{\jmath}$ (in standard position) as edges.

Solution. We have $-\hat{\imath} + 4\hat{\jmath} = [-1, 4] = [a_1, a_2]$ and $2\hat{\imath} + 3\hat{\jmath} = [2, 3] = [b_1, b_2]$ so the area of the parallelogram is $|\det(A)|$ where $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$. Hence the area is

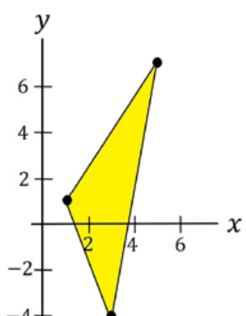
$$|\det(A)| = \left| \begin{vmatrix} -1 & 4 \\ 2 & 3 \end{vmatrix} \right| = |(-1)(3) - (4)(2)| = \boxed{11.}$$

□

Page 249 Number 26

Page 249 Number 26. Find the area of the triangle with vertices $(3, -4)$, $(1, 1)$, and $(5, 7)$.

Solution. First notice that:

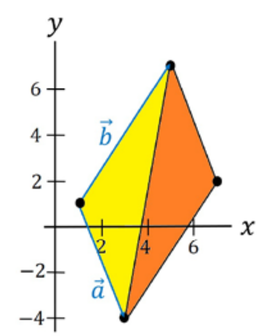


So we introduce vector $\vec{a}$ from $(1, 1)$ to $(3, -4)$ and vector $\vec{b}$ from $(1, 1)$ to $(5, 7)$, so that $\vec{a} = [a_1, a_2] = [(3) - (1), (-4) - (1)] = [2, -5]$ and $\vec{b} = [b_1, b_2] = [(5) - (1), (7) - (1) = [4, 6]$.

Page 249 Number 26 (continued)

Page 249 Number 26. Find the area of the triangle with vertices $(3, -4)$, $(1, 1)$, and $(5, 7)$.

Solution (continued). We can then find the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$ using the determinant; but this must be halved to find the area of the desired triangle:



So the area of the triangle is $\frac{1}{2}|\det(A)|$ where $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 4 & 6 \end{bmatrix}$. Then the area is

$$\frac{1}{2}|\det(A)| = \frac{1}{2} \left| \begin{vmatrix} 2 & -5 \\ 4 & 6 \end{vmatrix} \right| = \frac{1}{2}|(2)(6) - (-5)(4)| = \frac{1}{2}|32| = \boxed{16.}$$

□

Page 248 Number 16

Page 248 Number 16. Let $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 7\hat{k}$. Find $\vec{a} \times \vec{b}$.

Solution. We have $\vec{a} = [a_1, a_2, a_3] = [1, -1, 1]$ and $\vec{b} = [b_1, b_2, b_3] = [3, -2, 7]$. Then by definition,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} \\ &= \begin{vmatrix} -1 & 1 \\ -2 & 7 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 1 \\ 3 & 7 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} \hat{k} \\ &= ((-1)(7) - (1)(-2))\,\hat{\imath} - ((1)(7) - (1)(3))\,\hat{\jmath} \\ &\quad + ((1)(-2) - (-1)(3))\,\hat{k} \\ &= \boxed{-5\hat{\imath} - 4\hat{\jmath} + \hat{k} = [-5, -4, 1].}\end{aligned}$$

□

Theorem 4.1.A

Theorem 4.1.A. The area of the parallelogram determined by $\vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$ is $\|\vec{b} \times \vec{c}\|$.

Proof. We know from the first note of this section that the area squared is $A^2 = \|\vec{c}\|^2\|\vec{b}\|^2 - (\vec{c} \cdot \vec{b})^2$. In terms of components we have

$$A^2 = (c_1^2 + c_2^2 + c_3^2)(b_1^2 + b_2^2 + b_3^2) - (c_1b_1 + c_2b_2 + c_3b_3)^2.$$

Multiplying out we have

$$\begin{aligned}A^2 &= c_1^2b_1^2 + c_1^2b_2^2 + c_1^2b_3^2 + c_2^2b_1^2 + c_2^2b_2^2 + c_2^2b_3^2 + c_3^2b_1^2 + c_3^2b_2^2 + c_3^2b_3^2 \\ &\quad -(c_1^2b_1^2 + c_2^2b_2^2 + c_3^2b_3^2 + 2c_1b_1c_2b_2 + 2c_1b_1c_3b_3 + 2c_2b_2c_3b_3) \\ &= (c_3^2b_2^2 - 2c_2b_2c_3b_3 + c_2^2b_3^2) + (c_3^2b_1^2 - 2c_1b_1c_3b_3 + c_1^2b_3^2) \\ &\quad +(c_2^2b_1^2 - 2c_1b_1c_2b_2 + c_1^2b_2^2) \\ &= (b_2c_3 - b_3c_2)^2 + (b_1c_3 - b_3c_1)^2 + (b_1c_2 - b_2c_1)^2\end{aligned}$$

...

Theorem 4.1.A (continued)

Theorem 4.1.A. The area of the parallelogram determined by $\vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$ is $\|\vec{b} \times \vec{c}\|$.

Proof (continued). ...

$$\begin{aligned}A^2 &= (b_2c_3 - b_3c_2)^2 + (b_1c_3 - b_3c_1)^2 + (b_1c_2 - b_2c_1)^2 \\ &= \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}^2 + \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}^2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}^2 \\ &= \left\| \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \hat{k} \right\| = \|\vec{b} \times \vec{c}\|^2.\end{aligned}$$

Taking square roots we see that the claim is verified. □

Theorem 4.1.B

Theorem 4.1.B. The volume of a box determined by vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ is $V = |a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)| = |\vec{a} \cdot \vec{b} \times \vec{c}|$.

Proof. Consider the box determined by $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$:

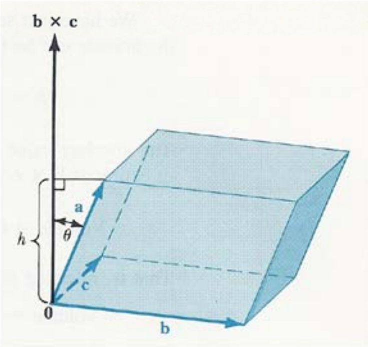


Figure 4.5, Page 244.

The volume of the box is the height times the area of the base.

Theorem 4.1.B (continued)

Theorem 4.1.B. The volume of a box determined by vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ is $V = |a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)| = |\vec{a} \cdot \vec{b} \times \vec{c}|$.

Proof (continued). The area of the base is $\|\vec{b} \times \vec{c}\|$ by Theorem 4.1.A. Now the height is

$$h = \|\vec{a}\| |\cos\theta| = \frac{\|\vec{b} \times \vec{c}\| \|\vec{a}\| |\cos\theta|}{\|\vec{b} \times \vec{c}\|} = \frac{|(\vec{b} \times \vec{c}) \cdot \vec{a}|}{\|\vec{b} \times \vec{c}\|}.$$

(Notice that if $\vec{b} \times \vec{c}$ is in the opposite direction as given in the illustration above, then θ would be greater than $\pi/2$ and $\cos\theta$ would be negative. Therefore the absolute value is necessary.) Therefore

$$V = (\text{Area of base})(\text{height}) = \|\vec{b} \times \vec{c}\| \frac{|(\vec{b} \times \vec{c}) \cdot \vec{a}|}{\|\vec{b} \times \vec{c}\|} = |(\vec{b} \times \vec{c}) \cdot \vec{a}|.$$

□

Page 249 Number 38

Page 249 Number 38. Find the volume of the box having the vectors (in standard position) $2\hat{\imath} + \hat{\jmath} - 4\hat{k}$, $3\hat{\imath} - \hat{\jmath} + 2\hat{k}$, $\hat{\imath} + 3\hat{\jmath} - 8\hat{k}$ as adjacent edges.

Solution. Let $\vec{a} = [a_1, a_2, a_3] = 2\hat{\imath} + \hat{\jmath} - 4k = [2, 1, -4]$, $\vec{b} = [b_1, b_2, b_3] = 3\hat{\imath} - \hat{\jmath} + 2\hat{k} = [3, -1, 2]$, and $\vec{c} = [c_1, c_2, c_3] = \hat{\imath} + 3\hat{\jmath} - 8\hat{k} = [1, 3, -8]$. Then by the previous Note, the volume of the box is $|\det(A)|$ where $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -4 \\ 3 & -1 & 2 \\ 1 & 3 & -8 \end{bmatrix}$. So the volume is

$$\left| \begin{vmatrix} 2 & 1 & -4 \\ 3 & -1 & 2 \\ 1 & 3 & -8 \end{vmatrix} \right| = \left| (2) \begin{vmatrix} -1 & 2 \\ 3 & -8 \end{vmatrix} - (1) \begin{vmatrix} 3 & 2 \\ 1 & -8 \end{vmatrix} + (-4) \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} \right|$$

$$= |2((-1)(-8) - (2)(3)) - ((3)(-8) - (2)(1)) - 4((3)(3) - (-1)(1))|$$

$$= |2(2) - (-26) - 4(10)| = \boxed{10.}$$ □

Page 249 Number 50

Page 249 Number 50. Use a determinant to ascertain whether the points $(0, 0, 0)$, $(2, 1, 1)$, $(3, -2, 1)$, $(-1, 2, 3)$ lie in a plane in $\mathbb{R}^3$.

Solution. We introduce three vectors determining a box. If the volume of the box is 0 then the points are in the same plane and if the volume is positive then the points are not coplanar. So we take vectors from $(0, 0, 0)$ to the other three points:
$\vec{a} = [a_1, a_2, a_3] = [(2) - (0), (1) - (0), (1) - (0)] = [2, 1, 1]$,
$\vec{b} = [b_1, b_2, b_3] = [(3) - (0), (-2) - (0), (1) - (0)] = [3, -2, 1]$,
$\vec{c} = [c_1, c_2, c_3] = [(-1) - (0), (2) - (0), (3) - (0)] = [-1, 2, 3]$. Then with $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$, the volume of the box is $|\det(A)|$:

Page 249 Number 50 (continued)

Page 249 Number 50. Use a determinant to ascertain whether the points $(0, 0, 0)$, $(2, 1, 1)$, $(3, -2, 1)$, $(-1, 2, 3)$ lie in a plane in $\mathbb{R}^3$.

Solution (continued). ...the volume of the box is $|\det(A)|$:

$$\left| \begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 2 & 3 \end{vmatrix} \right| = \left| (2) \begin{vmatrix} -2 & 1 \\ 2 & 3 \end{vmatrix} - (1) \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} + (1) \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} \right|$$

$$= |2((-2)(3) - (1)(2)) - ((3)(3) - (1)(-1)) + ((3)(2) - (-2)(-1))|$$

$$= |2(-8) - (10) + (4)| = 22.$$

Since the volume is not 0, then $\boxed{\text{the points do not lie in a plane in } \mathbb{R}^3.}$ □

Theorem 4.1

Theorem 4.1. Properties of Cross Product.
Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$.
(1) Anticommutivity: $\vec{b} \times \vec{c} = -\vec{c} \times \vec{b}$
(3) Distributive Properties: $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

Proof. (1) Page 247 Example 8. We have

$$\begin{aligned}
\vec{b} \times \vec{c} &= \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \hat{k} \\
&= (b_2c_3 - b_3c_2)\hat{\imath} - (b_1c_3 - b_3c_1)\hat{\jmath} + (b_1c_2 - b_2c_1)\hat{k} \\
&= -\left((b_3c_2 - b_2c_3)\hat{\imath} - (b_3c_1 - b_1c_3)\hat{\jmath} + (b_2c_1 - b_1c_2)\hat{k}\right) \\
&= -\left(\begin{vmatrix} c_2 & c_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} c_1 & c_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} c_1 & c_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} \right) \\
&= -\vec{c} \times \vec{b}
\end{aligned}$$

Theorem 4.1 (continued 1)

Theorem 4.1. Properties of Cross Product.
Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$.
(3) Distributive Properties: $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

Proof (continued). (3) Page 249 Number 58. We have

$$\vec{a} \times (\vec{b} + \vec{c}) = [a_1, a_2, a_3] \times ([b_1, b_2, b_3] + [c_1, c_2, c_3])$$

$$= [a_1, a_2, a_3] \times [b_1 + c_1, b_2 + c_2, b_3 + c_3] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 + c_2 & b_3 + c_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 + c_1 & b_3 + c_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 + c_1 & b_2 + c_2 \end{vmatrix} \hat{k}$$

Theorem 4.1 (continued 2)

Proof (continued). ...

$$\begin{aligned}
&= ((a_2)(b_3 + c_3) - (a_3)(b_2 + c_2))\,\hat{\imath} - ((a_1)(b_3 + c_3) - (a_3)(b_1 + c_1))\,\hat{\jmath} \\
&\quad + ((a_1)(b_2 + c_2) - (a_2)(b_1 + c_1))\,\hat{k} \\
&= (a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2)\hat{\imath} - (a_1b_3 + a_1c_3 - a_3b_1 - a_3c_1)\hat{\jmath} \\
&\quad + (a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1)\hat{k} \\
&= ((a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2))\,\hat{\imath} - ((a_1b_3 - a_3b_1) + (a_1c_3 - a_3c_1))\,\hat{\jmath} \\
&\quad + ((a_1b_2 - a_2b_1) + (a_1c_2 - a_2c_1))\,\hat{k} \\
&= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} \right) \hat{\imath} - \left(\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} \right) \hat{\jmath} \\
&\quad + \left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \right) \hat{k}
\end{aligned}$$

Theorem 4.1 (continued 3)

Proof (continued). ...

$$\begin{aligned}
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} + \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} - \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} \hat{\jmath} \\
&\quad + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \hat{k} \\
&= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} \right) \\
&\quad + \left(\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \hat{k} \right) \\
&= (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}).
\end{aligned}$$

□

Page 249 Number 56

Page 249 Number 56. Let $\vec{a}, \vec{b}, \vec{c}$ be vectors in $\mathbb{R}^3$. Simplify the expression $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$.

Solution. By Theorem 4.1(7), "Properties of Cross Products," $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$, and so $\vec{b} \times (\vec{c} \times \vec{a}) = (\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}$, $\vec{c} \times (\vec{a} \times \vec{b}) = (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}$. Hence

$$\begin{aligned}
& \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) \\
= & \left((\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}\right) + \left((\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}\right) + \left((\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}\right) \\
= & (\vec{a} \cdot \vec{c} - \vec{c} \cdot \vec{a})\vec{b} + (\vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b})\vec{c} + (\vec{c} \cdot \vec{b} - \vec{b} \cdot \vec{c})\vec{a} \\
= & 0\vec{b} + 0\vec{c} + 0\vec{a} \text{ since dot product is commutative by} \\
& \text{Theorem 1.3(D1), "Properties of Dot Products"} \\
= & \boxed{\vec{0}.}
\end{aligned}$$

□