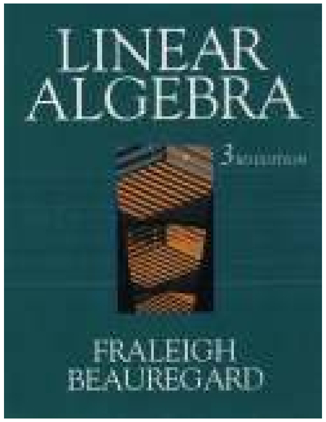
Linear Algebra

Chapter 4: Determinants

Section 4.2. The Determinant of a Square Matrix—Proofs of Theorems



Example 4.2.A.

Example 4.2.A. Find A_{11}, A_{12}, and A_{13} for

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Solution. To find A_{11}, we simply eliminate the first row and first column of A to get $A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$. Similarly, $A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ and $A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$. □

Page 262 Number 12

Page 262 Number 12. Find the cofactor of 3 in $A = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$.

Solution. We have $a_{21} = 3$, so we need $a'_{21} = (-1)^{2+1}\det(A_{21})$ where $A_{21} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is a minor matrix. So

$$a'_{21} = -\det(A_{21}) = -\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -((-1)(1) - (2)(2)) = \boxed{5.}$$ □

Example 4.2.B

Example 4.2.B. Find the determinant of $A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 4 & 0 & 1 & 4 \\ 1 & 0 & 2 & 1 \end{bmatrix}$.

Solution. We have
$\det(A) = a_{11}a'_{11} + a_{12}a'_{12} + a_{13}a'_{13} + a_{14}a'_{14} = 2a'_{11} + a'_{12} + a'_{14}$ where

$$\begin{aligned} a'_{11} &= (-1)^{1+1}\det(A_{11}) = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \end{vmatrix} \\ &= (2)\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} - (1)\begin{vmatrix} 0 & 4 \\ 0 & 1 \end{vmatrix} + (2)\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} \text{ by the} \\ &\quad \text{definition of determinant of a } 3 \times 3 \text{ matrix} \\ &= (2)((1)(1) - (4)(2)) - (1)((0)(1) - (4)(0)) + (2)((0)(2) - (1)(0)) \\ &= 2(-7) - 0 + 0 = -14, \end{aligned}$$

Example 4.2.B (continued 1)

Solution (continued).

$$\begin{aligned}
a'_{12} &= (-1)^{1+2}\det(A_{12}) = -\begin{vmatrix} 3 & 1 & 2 \\ 4 & 1 & 4 \\ 1 & 2 & 1 \end{vmatrix} \\
&= -\left((3)\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} - (1)\begin{vmatrix} 4 & 4 \\ 1 & 1 \end{vmatrix} + (2)\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix}\right) \\
&= -(3)\left((1)(1)-(4)(2)\right) + \left((4)(1)-(4)(1)\right) - 2\left((4)(2)-(1)(1)\right) \\
&= -3(-7) + (0) - 2(7) = 7, \\
a'_{14} &= (-1)^{1+4}\det(A_{14}) = -\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 1 & 0 & 2 \end{vmatrix} \\
&= -\left((3)\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} - (2)\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + (1)\begin{vmatrix} 4 & 0 \\ 1 & 0 \end{vmatrix}\right) \\
&= -(3)\left((0)(2)-(1)(0)\right) + (2)\left((4)(2)-(1)(1)\right) - \left((4)(0)-(0)(1)\right) \\
&= 0 + 2(7) - 0 = 14.
\end{aligned}$$

Example 4.2.B (continued 2)

Example 4.2.B. Find the determinant of $A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 4 & 0 & 1 & 4 \\ 1 & 0 & 2 & 1 \end{bmatrix}$.

Solution (continued). So

$$\begin{aligned}
\det(A) &= a_{11}a'_{11} + a_{12}a'_{12} + a_{13}a'_{13} + a_{14}a'_{14} \\
&= 2a'_{11} + a'_{12} + a'_{14} \\
&= 2(-14) + (7) + (14) \\
&= \boxed{-7.}
\end{aligned}$$

□

Example 4.2.C

Example 4.2.C. Find the determinant of $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 5 & 9 \\ 1 & 15 & 6 & 57 \end{bmatrix}$.

Solution. By Theorem 4.2, "General Expansion by Minors," we can find the determinant by expanding along any row or column, so we choose to start by expanding along the first column. We then have

$$\begin{aligned}
\det(A) &= (0) - (0) + (0) - (1)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 4 & 5 & 9 \end{vmatrix} \\
&= -\left((0) - (0) + (1)\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}\right) \text{ expanding along the first row} \\
&= -\left((0) - (0) + ((1)(5) - (2)(4))\right) = 3.
\end{aligned}$$

So $\boxed{\det(A) = 3.}$ □

Page 255 Example 4

Page 255 Example 4. Show that the determinant of an upper- or lower-triangular square matrix is the product of its diagonal elements.

Solution. Let

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1,n-1} & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2,n-1} & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3,n-1} & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

be an upper triangular matrix. By Theorem 4.2, "General Expansion by Minors," we calculate $\det(U)$ along the first column and then expand the determinant of each minor along the first column.

Page 255 Example 4 (continued 1)

Page 255 Example 4. Show that the determinant of an upper- or lower-triangular square matrix is the product of its diagonal elements.

Solution (continued). We get

$$\det(U) = \begin{vmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1,n-1} & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2,n-1} & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3,n-1} & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & u_{nn} \end{vmatrix}$$

$$= u_{11} \begin{vmatrix} u_{22} & u_{23} & \cdots & u_{2,n-1} & u_{2n} \\ 0 & u_{33} & \cdots & u_{3,n-1} & u_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & \cdots & 0 & u_{nn} \end{vmatrix} \cdots$$

Page 255 Example 4 (continued 2)

Page 255 Example 4. Show that the determinant of an upper- or lower-triangular square matrix is the product of its diagonal elements.

Solution (continued). ...

$$= u_{11}u_{22} \begin{vmatrix} u_{33} & u_{34} & \cdots & u_{3,n-1} & u_{3n} \\ 0 & u_{44} & \cdots & u_{4,n-1} & u_{4n} \\ \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & \cdots & 0 & u_{nn} \end{vmatrix} = u_{11}u_{22}u_{33}\cdots u_{nn}.$$

That is, $\det(U) = u_{11}u_{22}u_{33}\cdots u_{nn}$, as claimed. □

Theorem 4.2.A

Theorem 4.2.A. Properties of the Determinant.
Let A be a square matrix.

1. The Transpose Property: $\det(A) = \det(A^T)$.
2. The Row-Interchange Property: If two different rows of a square matrix A are interchanged, the determinant of the resulting matrix is $-\det(A)$.
3. The Equal-Rows Property: If two rows of a square matrix A are equal, then $\det(A) = 0$.
4. The Scalar-Multiplication Property: If a single row of a square matrix A is multiplied by a scalar r, the determinant of the resulting matrix if $r\det(A)$.
5. The Row-Addition Property: If the product of one row of A by a scalar r is added to a different row of A, the determinant of the resulting matrix is the same as $\det(A)$.

Theorem 4.2.A(1), The Transpose Property

Theorem 4.2.A. Properties of the Determinant.
Let A be a square matrix.

1. The Transpose Property: $\det(A) = \det(A^T)$.

Proof. (1) The result vacuously holds for a 1×1 matrix. For 2×2 matrix $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ we have $\det(A) = (a_1)(b_2) - (a_2)(b_1)$, $A^T = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, and $\det(A^T) = (a_1)(b_2) - (b_1)(a_2)$; hence the result holds for all 2×2 matrices. We use mathematical induction (see Appendix A). Assume the property holds for all matrices of size $k \times j$ for $k = 1, 2, \ldots, n-1$. We will prove that this shows that the result holds for $k = n$ (that is, for $n \times n$ matrices) and then the claim holds by induction.

Theorem 4.2.A(1) (continued)

Proof (continued). Let A be an $n \times n$ matrix. Then by Definition 4.1, "Cofactors and Determinants," we have

$$\det(A) = a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{n+1}a_{1n}|A_{1n}|.$$

With $B = A^T$ we have that $a_{1j} = b_{j1}$ and $A_{1j}^T = B_{j1}$. So applying Theorem 4.2, "General Expansion by Minors," we can compute $\det(B)$ by expanding along the first column of B to get

$$\begin{aligned}
\det(A^T) &= \det(B) = b_{11}|B_{11}| - b_{21}|B_{21}| + \cdots + (-1)^{n+1}b_{n1}|B_{n1}| \\
&= a_{11}|A_{11}^T| - a_{12}|A_{12}^T| + \cdots + (-1)^{n+1}a_{1n}|A_{1n}^T| \\
&\quad \text{since } a_{1j} = b_{j1} \text{ and } A_{1j}^T = B_{j1} \\
&= a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{n+1}a_{1n}|A_{1n}| \\
&\quad \text{since } A_{1j} \text{ is } (n-1)\times(n-1) \text{ and so,} \\
&\quad \text{by the induction hypothesis, } |A_{1j}{}^T| = |B_{j1}| \\
&= \det(A).
\end{aligned}$$

Theorem 4.2.A(2), The Row-Interchange Property

Theorem 4.2.A. Properties of the Determinant.
Let A be a square matrix.

2. The Row-Interchange Property: If two different rows of a square matrix A are interchanged, the determinant of the resulting matrix is $-\det(A)$.

Proof (continued). So the result holds for $k = n$. Therefore, by mathematical induction, (1) holds for all $n \times n$ matrices where n is a natural number.

(2) We again use mathematical induction. For $n = 2$, we have

$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = (b_1)(a_2) - (b_2)(a_1) = -((a_1)(b_2) - (a_2)(b_1)) = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

so the result holds for $n = 2$. Assume the property holds for all matrices of size $k \times k$ for $k = 1, 2, \ldots, n-1$. Let A be an $n \times n$ matrix and let B be the matrix obtained from A by interchanging the ith row and the rth row.

Theorem 4.2.A(2) (continued)

Proof (continued). Since $n > 2$, we can choose a kth row for expansion by minors, where $k \notin \{r, i\}$. Consider the cofactors

$$(-1)^{k+j}|A_{kj}| \text{ and } (-1)^{k+j}|B_{kj}|.$$

These numbers must have opposite signs, by our induction hypothesis, since the minor matrices A_{kj} and B_{kj} have size $(n-1) \times (n-1)$, and B_{kj} can be obtained from A_{kj} by interchanging two rows (namely, the ith and rth rows). That is, $|B_{kj}| = -|A_{kj}|$ and so $b'_{kj} = -a'_{kj}$. So applying Theorem 4.2, "General Expansion by Minors," we can compute $\det(B)$ by expanding along the kth row of A and the kth row of B we find:

$$\begin{aligned}
\det(A) &= a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn} \\
&= b_{k1}a'_{k1} + b_{k2}a'_{k2} + \cdots + b_{kn}a'_{kn} \\
&\quad \text{since the } k\text{th row of } A \text{ is the same as the } k\text{th row of } B \\
&= b_{k1}(-b'_{k1}) + b_{k2}(-b'_{k2}) + \cdots + b_{kn}(-b'_{kn}) \text{ since } b'_{kj} = -a'_{kj} \\
&= -(b_{k1}b'_{k1} + b_{k2}b'_{k2} + \cdots + b_{kn}b'_{kn}) = -\det(B).
\end{aligned}$$

Theorem 4.2.A(3), The Equal-Rows Property

Theorem 4.2.A. Properties of the Determinant.
Let A be a square matrix.

3. The Equal-Rows Property: If two rows of a square matrix A are equal, then $\det(A) = 0$.

Proof (continued). So the result holds for $k = n$. Therefore, by mathematical induction, (2) holds for all $n \times n$ matrices where n is a natural number.

(3) Let B be the matrix obtained from A by interchanging the two equal rows (so $B = A$). By the Row-Interchange Property, $\det(B) = -\det(A)$. But since $B = A$, this implies $\det(B) = \det(A)$. Hence $\det(A) = -\det(A)$ and we must have $\det(A) = 0$.

Theorem 4.2.A(4), The Scalar-Multiplication Property

Theorem 4.2.A. Properties of the Determinant.
Let A be a square matrix.

4. The Scalar-Multiplication Property: If a single row of a square matrix A is multiplied by a scalar r, the determinant of the resulting matrix if $r\det(A)$.

Proof (continued). (4) Let $r \in \mathbb{R}$ be a scalar and let B be the matrix obtained from A by multiplying the kth row of A by r; so the kth row of B is $[ra_{k1}, ra_{k2}, \ldots, ra_{kn}]$ so $b_{kj} = ra_{kj}$ for $j = 1, 2, \ldots, n$. Using Theorem 4.2, "General Expansion by Minors," we can compute $\det(A)$ by expanding along the kth row of A to $\det(A)$ get in terms of cofactors that

$$\det(A) = a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn}.$$

Since all rows of B equal the corresponding rows of A, except for the kth row, then the minors satisfy $A_{kj} = B_{kj}$ and the cofactors satisfy $a'_{kj} = b'_{kj}$ for $j = 1, 2, \ldots, n$.

Theorem 4.2.A(4), The Scalar-Multiplication Property (continued)

Theorem 4.2.A. Properties of the Determinant.
Let A be a square matrix.

4. The Scalar-Multiplication Property: If a single row of a square matrix A is multiplied by a scalar r, the determinant of the resulting matrix if $r\det(A)$.

Proof (continued). Finding $\det(B)$ by expanding along the kth row gives

$$\begin{aligned} \det(B) &= b_{k1}b'_{k1} + b_{k2}b'_{k2} + \cdots + b_{kn}b'_{kn} \\ &= ra_{k1}a'_{k1} + ra_{k2}a'_{k2} + \cdots + ra_{kn}a'_{kn} \\ &\quad \text{since } b_{kj} = ra_{kj} \text{ and } a'_{kj} = b'_{kj} \\ &= r(a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn}) = r\det(A). \end{aligned}$$

So the result holds for $k = n$. Therefore, by mathematical induction, (4) holds for all $n \times n$ matrices where n is a natural number.

Theorem 4.2.A(5), The Row-Addition Property

Theorem 4.2.A. Properties of the Determinant.
Let A be a square matrix.

5. The Row-Addition Property: If the product of one row of A by a scalar r is added to a different row of A, the determinant of the resulting matrix is the same as $\det(A)$.

Proof (continued). (5) The ith row of A is $[a_{i1}, a_{i2}, \ldots, a_{in}]$ and the kth row of A is $[a_{k1}, a_{k2}, \ldots, a_{kn}]$ where $i \neq k$. So if B is obtained from A by adding r times Row i to Row k, that is $[ra_{i1} + a_{k1}, ra_{i2} + a_{k2}, \ldots, ra_{in} + a_{kn}]$. As in the proof of Property 4, the minors satisfy $A_{kj} = B_{kj}$ and the cofactors satisfy $a'_{kj} = b'_{kj}$ for $j = 1, 2, \ldots, n$.

Theorem 4.2.A(5) The Row-Addition Property (continued)

Proof (continued). Using Theorem 4.2, "General Expansion by Minors," and expanding all determinant along the kth row, we have

$$\begin{aligned} \det(B) &= b_{k1}b'_{k1} + b_{k2}b'_{k2} + \cdots + b_{kn}b'_{kn} \\ &= (ra_{i1} + a_{k1})a'_{k1} + (ra_{i2} + a_{k2})a'_{k2} + \cdots + (ra_{in} + a_{kn})a'_{kn} \\ &\quad \text{since } b_{kj} = ra_{ij} + a_{kj} \text{ and } b'_{kj} = a_{kj} \\ &= r(a_{i1}a'_{k1} + a_{i2}a'_{k2} + \cdots + a_{in}a'_{kn}) \\ &\quad +(a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn}) \\ &= r\det(C) + \det(A) \end{aligned}$$

where matrix C is an $n \times n$ matrix with the same rows as matrix A, except that the kth row of C is the same as the ith row of A. Since $i \neq k$, then Row i and Row k of C are the same and so by Property 3, $\det(C) = 0$. Therefore, $\det(B) = \det(A)$, as claimed. □

Page 261 Number 8

Page 261 Number 8. Use row reduction and Theorem 4.2.A to find

$$\det(A) \text{ for } A = \begin{bmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix}.$$

Solution. Row reducing we have

$$A = \begin{bmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix} \underset{R_4 \to R_4 - 2R_1}{\overset{R_2 \to R_2 - 3R_1}{\widetilde{R_3 \to R_3 - 4R_1}}} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & -2 & 5 & -28 \\ 0 & 1 & 2 & -12 \end{bmatrix}$$

$$\underset{R_4 \to R_4 - R_2}{\widetilde{R_3 \to R_3 + 2R_2}} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & 11 & -62 \\ 0 & 0 & -1 & 5 \end{bmatrix} \widetilde{R_3 \leftrightarrow R_4} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 11 & -62 \end{bmatrix}$$

Page 261 Number 8 (continued)

Page 261 Number 8. Use row reduction and Theorem 4.2.A to find

$$\det(A) \text{ for } A = \begin{bmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix}.$$

Solution (continued). ...

$$\begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 11 & -62 \end{bmatrix} \widetilde{R_4 \to R_4 + 11R_3} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & -7 \end{bmatrix} = H.$$

So $A \sim H$ through a sequence of 6 row-additions and one row-interchange. Hence, by Theorem 4.2.A (Properties 2 and 5) $\det(A) = -\det(H)$. Now H is upper-triangular, so as shown in Page 255 Example 4, $\det(H) = (2)(1)(-1)(-7) = 14$. Hence, $\boxed{\det(A) = -\det(H) = -14.}$ □

Theorem 4.3

Theorem 4.3. Determinant Criterion for Invertibility.
A square matrix A is invertible if and only if $\det(A) \neq 0$. Equivalently, A is singular if and only if $\det(A) = 0$.

Proof. As commented above, A can be reduced to an echelon form H without multiplying rows by scalars (i.e., "row scaling") so $\det(A) = \pm\det(H)$. The H is upper triangular and so by Page 255 Example 4, the determinant of A is the product of its diagonal entries. Now A is invertible if and only if A has only nonzero entries on its main diagonal since by Theorem 1.12, "Conditions for A^{-1} to Exist," A is invertible if and only if it is row equivalent to $\mathcal{I}$. So $\det(A) = \pm\det(H) \neq 0$ if and only if A is invertible. □

Theorem 4.4

Theorem 4.4. The Multiplicative Property.
If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Proof. First, if A is a diagonal matrix then

$$AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1n} \\ a_{22}b_{21} & a_{22}b_{22} & \cdots & a_{22}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn}b_{n1} & a_{nn}b_{n2} & \cdots & a_{nn}b_{nn} \end{bmatrix}$$

and so by Theorem 4.2.A(4), "The Scalar-Multiplication Property," $\det(AB) = a_{11}a_{22}\cdots a_{nn}\det(B) = \det(A)\det(B)$ because A upper triangular implies $\det(A) = a_{11}a_{22}\cdots a_{nn}$ by Page 255 Example 4.

Theorem 4.4 (continued 1)

Proof (continued). Second, if A is not invertible then AB is not invertible by Exercise 30, so by Theorem 4.1, "Determinant Criterion for Invertibility," $\det(A) = \det(AB) = 0$.

Third, for A invertible then as seen in the proof of Theorem 4.3, "Determinant Criterion for Invertibility," A can be row reduced through row-interchange and row-addition elementary row operations to an upper-triangular matrix with nonzero entries on the diagonal. We can then use row-interchange and row-addition, as we would in the Gauss-Jordan Method, to reduce A to a diagonal matrix where no diagonal entries are 0. So there is a matrix E, a product of elementary matrices corresponding to row-interchange and row-addition, such that $D = EA$ where D is the diagonal matrix just described. Then Theorem 4.2.A(2) and (5), "Row-Interchange Property" and "Row-Addition Property," imply that $\det(A) = (-1)^r \det(D)$ where r is the number of row interchanges used in the row reduction of A to D.

Theorem 4.4 (continued 2)

Theorem 4.4. The Multiplicative Property.
If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Proof (continued). The same sequence of elementary row operations will reduce the matrix AB to the matrix $E(AB) = (EA)B = DB$. So, similar to the determinant of A, we have $\det(AB) = (-1)^r \det(DB)$. Therefore,

$$\begin{aligned}
\det(AB) &= (-1)^r \det(DB) \\
&= (-1)^r \det(D)\det(B) \text{ since we showed that} \\
&\quad\ \text{the theorem holds if the first matrix is diagonal} \\
&= ((-1)^r \det(D))\det(B) \\
&= \det(A)\det(B) \text{ since } \det(A) = (-1)^r \det(D).
\end{aligned}$$

Hence, $\det(AB) = \det(A)\det(B)$ in general. □

Page 262 Number 28

Page 262 Number 28. Find the values of λ for which the matrix
$A = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 4 \\ 0 & 1 & 1-\lambda \end{bmatrix}$ is singular.

Solution. By Theorem 4.3, "Determinant Criterion for Invertibility," A is singular if and only if $\det(A) = 0$. We have by the definition of determinant of a 3×3 matrix that

$$\begin{aligned}
\det(A) &= (2-\lambda)\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} - (0) + (0) \\
&= (2-\lambda)\left((1-\lambda)(1-\lambda) - (4)(1)\right) = (2-\lambda)(1 - 2\lambda + \lambda^2 - 4) \\
&= (2-\lambda)(\lambda^2 - 2\lambda - 3) = (2-\lambda)(\lambda-3)(\lambda+1).
\end{aligned}$$

So $\det(A) = 0$ if and only if $\lambda = -1$, $\lambda = 2$, or $\lambda = 3$. That is, A is singular if and only if $\boxed{\lambda = -1, \lambda = 2, \text{ or } \lambda = 3.}$
Note. We'll see why this type of problem is of interest in the next chapter.
□

Page 262 Number 30

Page 262 Number 30. If A and B are $n \times n$ matrices and if A is singular, prove (without using Theorem 4.4) that AB is also singular.

Solution. We give a proof by contradiction. Let A be singular and ASSUME that AB is nonsingular. Then there is $(AB)^{-1}$ where $(AB)(AB)^{-1} = \mathcal{I}$. But then by Theorem 1.3.A, "Associativity of Matrix Multiplication," $A(B(AB)^{-1}) = \mathcal{I}$ and so $B(AB)^{-1}$ is the inverse of A (by Theorem 1.11, "A Commutative Property," we have $(B(AB)^{-1})A = \mathcal{I}$ also). But this CONTRADICTS the hypothesis that A is singular. So the assumption that AB is nonsingular is false and hence AB is singular, as claimed. □

Page 262 Number 32

Page 262 Number 32. If A and C are $n \times n$ matrices with C invertible, prove that $\det(A) = \det(C^{-1}AC)$. HINT: By Exercise 31, for invertible C we have $\det(C^{-1}) = 1/\det(C)$.

Prove. By Theorem 4.4, "The Multiplicative Property,"

$$\det(C^{-1}AC) = \det(C^{-1}(AC)) = \det(C^{-1})\det(AC) = \det(C^{-1})\det(A)\det(C)$$

By Exercise 31, $\det(C^{-1}) = 1/\det(C)$, so

$$\det(C^{-1}AC) = (1/\det(C))\det(A)\det(C) = \det(A),$$

as claimed. □