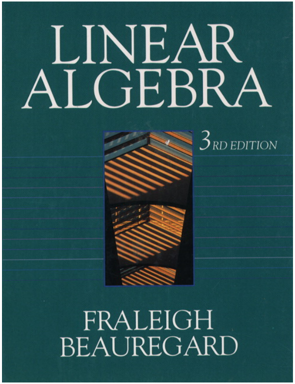
# Linear Algebra

**Chapter 4: Determinants**

Section 4.2. The Determinant of a Square Matrix—Proofs of Theorems



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# Example 4.2.A.

**Example 4.2.A.** Find $A_{11}$, $A_{12}$, and $A_{13}$ for

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

**Solution.** To find $A_{11}$, we simply eliminate the first row and first column of $A$ to get $A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$. Similarly, $A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ and $A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$. □

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# Page 262 Number 12

**Page 262 Number 12.** Find the cofactor of 3 in $A = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$.

**Solution.** We have $a_{21} = 3$, so we need $a'_{21} = (-1)^{2+1}\det(A_{21})$ where $A_{21} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is a minor matrix. So

$$a'_{21} = -\det(A_{21}) = -\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -((-1)(1) - (2)(2)) = \boxed{5.} \quad \square$$

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# Example 4.2.B

**Example 4.2.B.** Find the determinant of $A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 4 & 0 & 1 & 4 \\ 1 & 0 & 2 & 1 \end{bmatrix}$.

**Solution.** We have

$\det(A) = a_{11}a'_{11} + a_{12}a'_{12} + a_{13}a'_{13} + a_{14}a'_{14} = 2a'_{11} + a'_{12} + a'_{14}$ where

$$a'_{11} = (-1)^{1+1}\det(A_{11}) = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \end{vmatrix}$$

# Example 4.2.B

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$$a'_{11} = (-1)^{1+1}\det(A_{11}) = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= (2)\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} - (1)\begin{vmatrix} 0 & 4 \\ 0 & 1 \end{vmatrix} + (2)\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}$$ by the definition of determinant of a $3 \times 3$ matrix

$$= (2)\left((1)(1) - (4)(2)\right) - (1)\left((0)(1) - (4)(0)\right) + (2)\left((0)(2) - (1)(0)\right)$$

$$= 2(-7) - 0 + 0 = -14,$$

# Example 4.2.B

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$$\begin{aligned} a'_{11} &= (-1)^{1+1}\det(A_{11}) = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \end{vmatrix} \\ &= (2)\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} - (1)\begin{vmatrix} 0 & 4 \\ 0 & 1 \end{vmatrix} + (2)\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} \text{ by the} \\ &\quad \text{definition of determinant of a } 3 \times 3 \text{ matrix} \\ &= (2)\left((1)(1) - (4)(2)\right) - (1)\left((0)(1) - (4)(0)\right) + (2)\left((0)(2) - (1)(0)\right) \\ &= 2(-7) - 0 + 0 = -14, \end{aligned}$$

# Example 4.2.B (continued 1)

**Solution (continued).**

$$\begin{aligned}
a'_{12} &= (-1)^{1+2}\det(A_{12}) = -\begin{vmatrix} 3 & 1 & 2 \\ 4 & 1 & 4 \\ 1 & 2 & 1 \end{vmatrix} \\
&= -\left( (3)\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} - (1)\begin{vmatrix} 4 & 4 \\ 1 & 1 \end{vmatrix} + (2)\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} \right) \\
&= -(3)\,((1)(1)-(4)(2)) + ((4)(1)-(4)(1)) - 2\,((4)(2)-(1)(1)) \\
&= -3(-7) + (0) - 2(7) = 7,
\end{aligned}$$

$$\begin{aligned}
a'_{14} &= (-1)^{1+4}\det(A_{14}) = -\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 1 & 0 & 2 \end{vmatrix} \\
&= -\left( (3)\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} - (2)\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + (1)\begin{vmatrix} 4 & 0 \\ 1 & 0 \end{vmatrix} \right) \\
&= -(3)\,((0)(2)-(1)(0)) + (2)\,((4)(2)-(1)(1)) - ((4)(0)-(0)(1)) \\
&= 0 + 2(7) - 0 = 14.
\end{aligned}$$

# Example 4.2.B (continued 1)

**Solution (continued).**

$$\begin{aligned}
a'_{12} &= (-1)^{1+2}\det(A_{12}) = -\begin{vmatrix} 3 & 1 & 2 \\ 4 & 1 & 4 \\ 1 & 2 & 1 \end{vmatrix} \\
&= -\left( (3)\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} - (1)\begin{vmatrix} 4 & 4 \\ 1 & 1 \end{vmatrix} + (2)\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} \right) \\
&= -(3)\left((1)(1)-(4)(2)\right) + \left((4)(1)-(4)(1)\right) - 2\left((4)(2)-(1)(1)\right) \\
&= -3(-7) + (0) - 2(7) = 7, \\
a'_{14} &= (-1)^{1+4}\det(A_{14}) = -\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 1 & 0 & 2 \end{vmatrix} \\
&= -\left( (3)\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} - (2)\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + (1)\begin{vmatrix} 4 & 0 \\ 1 & 0 \end{vmatrix} \right) \\
&= -(3)\left((0)(2)-(1)(0)\right) + (2)\left((4)(2)-(1)(1)\right) - \left((4)(0)-(0)(1)\right) \\
&= 0 + 2(7) - 0 = 14.
\end{aligned}$$

# Example 4.2.B (continued 2)

**Example 4.2.B.** Find the determinant of $A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 4 & 0 & 1 & 4 \\ 1 & 0 & 2 & 1 \end{bmatrix}$.

**Solution (continued).** So

$$\begin{aligned} \det(A) &= a_{11}a'_{11} + a_{12}a'_{12} + a_{13}a'_{13} + a_{14}a'_{14} \\ &= 2a'_{11} + a'_{12} + a'_{14} \\ &= 2(-14) + (7) + (14) \\ &= \boxed{-7.} \end{aligned}$$

□

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**Example 4.2.B.** Find the determinant of $A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 4 & 0 & 1 & 4 \\ 1 & 0 & 2 & 1 \end{bmatrix}$.

**Solution (continued).** So

$$\begin{aligned} \det(A) &= a_{11}a'_{11} + a_{12}a'_{12} + a_{13}a'_{13} + a_{14}a'_{14} \\ &= 2a'_{11} + a'_{12} + a'_{14} \\ &= 2(-14) + (7) + (14) \\ &= \boxed{-7.} \end{aligned}$$

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# Example 4.2.C

**Example 4.2.C.** Find the determinant of $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 5 & 9 \\ 1 & 15 & 6 & 57 \end{bmatrix}$.

**Solution.** By Theorem 4.2, "General Expansion by Minors," we can find the determinant by expanding along any row or column, so we choose to start by expanding along the first column.

# Example 4.2.C

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$$\begin{aligned} \det(A) &= (0) - (0) + (0) - (1)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 4 & 5 & 9 \end{vmatrix} \\ &= -\left((0) - (0) + (1)\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}\right) \text{ expanding along the first row} \\ &= -((0) - (0) + ((1)(5) - (2)(4))) = 3. \end{aligned}$$

So $\boxed{\det(A) = 3.}$ □

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So $\boxed{\det(A) = 3.}$ □

# Page 255 Example 4

**Page 255 Example 4.** Show that the determinant of an upper- or lower-triangular square matrix is the product of its diagonal elements.

**Solution.** Let

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1,n-1} & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2,n-1} & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3,n-1} & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

be an upper triangular matrix.

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be an upper triangular matrix. By Theorem 4.2, "General Expansion by Minors," we calculate $\det(U)$ along the first column and then expand the determinant of each minor along the first column.

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**Solution (continued).** We get

$$\det(U) = \begin{vmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1,n-1} & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2,n-1} & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3,n-1} & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & u_{nn} \end{vmatrix}$$

$$= u_{11} \begin{vmatrix} u_{22} & u_{23} & \cdots & u_{2,n-1} & u_{2n} \\ 0 & u_{33} & \cdots & u_{3,n-1} & u_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & \cdots & 0 & u_{nn} \end{vmatrix} \cdots$$

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$$= u_{11} \begin{vmatrix} u_{22} & u_{23} & \cdots & u_{2,n-1} & u_{2n} \\ 0 & u_{33} & \cdots & u_{3,n-1} & u_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & \cdots & 0 & u_{nn} \end{vmatrix} \cdots$$

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**Page 255 Example 4.** Show that the determinant of an upper- or lower-triangular square matrix is the product of its diagonal elements.

**Solution (continued).** ...

$$= u_{11}u_{22}\begin{vmatrix} u_{33} & u_{34} & \cdots & u_{3,n-1} & u_{3n} \\ 0 & u_{44} & \cdots & u_{4,n-1} & u_{4n} \\ \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & \cdots & 0 & u_{nn} \end{vmatrix} = u_{11}u_{22}u_{33}\cdots u_{nn}.$$

That is, $\det(U) = u_{11}u_{22}u_{33}\cdots u_{nn}$, as claimed. □

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That is, $\det(U) = u_{11}u_{22}u_{33}\cdots u_{nn}$, as claimed. □

# Theorem 4.2.A

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

1. The Transpose Property: $\det(A) = \det(A^T)$.
2. The Row-Interchange Property: If two different rows of a square matrix $A$ are interchanged, the determinant of the resulting matrix is $-\det(A)$.
3. The Equal-Rows Property: If two rows of a square matrix $A$ are equal, then $\det(A) = 0$.
4. The Scalar-Multiplication Property: If a single row of a square matrix $A$ is multiplied by a scalar $r$, the determinant of the resulting matrix if $r\det(A)$.
5. The Row-Addition Property: If the product of one row of $A$ by a scalar $r$ is added to a different row of $A$, the determinant of the resulting matrix is the same as $\det(A)$.

# Theorem 4.2.A(1), The Transpose Property

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

1. The Transpose Property: $\det(A) = \det(A^T)$.

**Proof. (1)** The result vacuously holds for a $1 \times 1$ matrix. For $2 \times 2$ matrix $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ we have $\det(A) = (a_1)(b_2) - (a_2)(b_1)$, $A^T = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, and $\det(A^T) = (a_1)(b_2) - (b_1)(a_2)$; hence the result holds for all $2 \times 2$ matrices. We use mathematical induction (see Appendix A). Assume the property holds for all matrices of size $k \times j$ for $k = 1, 2, \ldots, n-1$. We will prove that this shows that the result holds for $k = n$ (that is, for $n \times n$ matrices) and then the claim holds by induction.

# Theorem 4.2.A(1), The Transpose Property

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# Theorem 4.2.A(1) (continued)

**Proof (continued).** Let $A$ be an $n \times n$ matrix. Then by Definition 4.1, "Cofactors and Determinants," we have

$$\det(A) = a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{n+1} a_{1n}|A_{1n}|.$$

With $B = A^T$ we have that $a_{1j} = b_{j1}$ and $A^T_{1j} = B_{j1}$. So applying Theorem 4.2, "General Expansion by Minors," we can compute $\det(B)$ by expanding along the first column of $B$ to get

$$\begin{aligned} \det(A^T) &= \det(B) = b_{11}|B_{11}| - b_{21}|B_{21}| + \cdots + (-1)^{n+1} b_{n1}|B_{n1}| \\ &= a_{11}|A^T_{11}| - a_{12}|A^T_{12}| + \cdots + (-1)^{n+1} a_{1n}|A^T_{1n}| \\ &\quad \text{since } a_{1j} = b_{j1} \text{ and } A^T_{1j} = B_{j1} \end{aligned}$$

# Theorem 4.2.A(1) (continued)

**Proof (continued).** Let $A$ be an $n \times n$ matrix. Then by Definition 4.1, "Cofactors and Determinants," we have

$$\det(A) = a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{n+1} a_{1n}|A_{1n}|.$$

With $B = A^T$ we have that $a_{1j} = b_{j1}$ and $A_{1j}^T = B_{j1}$. So applying Theorem 4.2, "General Expansion by Minors," we can compute $\det(B)$ by expanding along the first column of $B$ to get

$$\begin{aligned} \det(A^T) &= \det(B) = b_{11}|B_{11}| - b_{21}|B_{21}| + \cdots + (-1)^{n+1} b_{n1}|B_{n1}| \\ &= a_{11}|A_{11}^T| - a_{12}|A_{12}^T| + \cdots + (-1)^{n+1} a_{1n}|A_{1n}^T| \\ &\quad \text{since } a_{1j} = b_{j1} \text{ and } A_{1j}^T = B_{j1} \\ &= a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{n+1} a_{1n}|A_{1n}| \\ &\quad \text{since } A_{1j} \text{ is } (n-1) \times (n-1) \text{ and so,} \\ &\quad \text{by the induction hypothesis, } |{A_{1j}}^T| = |B_{j1}| \\ &= \det(A). \end{aligned}$$

# Theorem 4.2.A(1) (continued)

**Proof (continued).** Let $A$ be an $n \times n$ matrix. Then by Definition 4.1, "Cofactors and Determinants," we have

$$\det(A) = a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{n+1} a_{1n}|A_{1n}|.$$

With $B = A^T$ we have that $a_{1j} = b_{j1}$ and $A_{1j}^T = B_{j1}$. So applying Theorem 4.2, "General Expansion by Minors," we can compute $\det(B)$ by expanding along the first column of $B$ to get

$$\begin{aligned}
\det(A^T) &= \det(B) = b_{11}|B_{11}| - b_{21}|B_{21}| + \cdots + (-1)^{n+1} b_{n1}|B_{n1}| \\
&= a_{11}|A_{11}^T| - a_{12}|A_{12}^T| + \cdots + (-1)^{n+1} a_{1n}|A_{1n}^T| \\
&\quad \text{since } a_{1j} = b_{j1} \text{ and } A_{1j}^T = B_{j1} \\
&= a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{n+1} a_{1n}|A_{1n}| \\
&\quad \text{since } A_{1j} \text{ is } (n-1) \times (n-1) \text{ and so,} \\
&\quad \text{by the induction hypothesis, } |{A_{1j}}^T| = |B_{j1}| \\
&= \det(A).
\end{aligned}$$

# Theorem 4.2.A(2), The Row-Interchange Property

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

2. The Row-Interchange Property: If two different rows of a square matrix $A$ are interchanged, the determinant of the resulting matrix is $-\det(A)$.

**Proof (continued).** So the result holds for $k = n$. Therefore, by mathematical induction, (1) holds for all $n \times n$ matrices where $n$ is a natural number.

**(2)** We again use mathematical induction. For $n = 2$, we have

$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = (b_1)(a_2) - (b_2)(a_1) = -((a_1)(b_2) - (a_2)(b_1)) = - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

so the result holds for $n = 2$.

# Theorem 4.2.A(2), The Row-Interchange Property

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Let $A$ be a square matrix.

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**(2)** We again use mathematical induction. For $n = 2$, we have

$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = (b_1)(a_2) - (b_2)(a_1) = -((a_1)(b_2) - (a_2)(b_1)) = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

so the result holds for $n = 2$. Assume the property holds for all matrices of size $k \times k$ for $k = 1, 2, \ldots, n-1$. Let $A$ be an $n \times n$ matrix and let $B$ be the matrix obtained from $A$ by interchanging the $i$th row and the $r$th row.

# Theorem 4.2.A(2), The Row-Interchange Property

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

2. The Row-Interchange Property: If two different rows of a square matrix $A$ are interchanged, the determinant of the resulting matrix is $-\det(A)$.

**Proof (continued).** So the result holds for $k = n$. Therefore, by mathematical induction, (1) holds for all $n \times n$ matrices where $n$ is a natural number.

**(2)** We again use mathematical induction. For $n = 2$, we have

$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = (b_1)(a_2) - (b_2)(a_1) = -((a_1)(b_2) - (a_2)(b_1)) = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

so the result holds for $n = 2$. Assume the property holds for all matrices of size $k \times k$ for $k = 1, 2, \ldots, n-1$. Let $A$ be an $n \times n$ matrix and let $B$ be the matrix obtained from $A$ by interchanging the $i$th row and the $r$th row.

# Theorem 4.2.A(2) (continued)

**Proof (continued).** Since $n > 2$, we can choose a $k$th row for expansion by minors, where $k \notin \{r, i\}$. Consider the cofactors

$$(-1)^{k+j}|A_{kj}| \text{ and } (-1)^{k+j}|B_{kj}|.$$

These numbers must have opposite signs, by our induction hypothesis, since the minor matrices $A_{kj}$ and $B_{kj}$ have size $(n-1) \times (n-1)$, and $B_{kj}$ can be obtained from $A_{kj}$ by interchanging two rows (namely, the $i$th and $r$th rows). That is, $|B_{kj}| = -|A_{kj}|$ and so $b'_{kj} = -a'_{kj}$.

# Theorem 4.2.A(2) (continued)

**Proof (continued).** Since $n > 2$, we can choose a $k$th row for expansion by minors, where $k \notin \{r, i\}$. Consider the cofactors

$$(-1)^{k+j}|A_{kj}| \text{ and } (-1)^{k+j}|B_{kj}|.$$

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$$\begin{aligned}
\det(A) &= a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn} \\
&= b_{k1}a'_{k1} + b_{k2}a'_{k2} + \cdots + b_{kn}a'_{kn} \\
&\quad \text{ since the } k\text{th row of } A \text{ is the same as the } k\text{th row of } B \\
&= b_{k1}(-b'_{k1}) + b_{k2}(-b'_{k2}) + \cdots + b_{kn}(-b'_{kn}) \text{ since } b'_{kj} = -a'_{kj} \\
&= -(b_{k1}b'_{k1} + b_{k2}b'_{k2} + \cdots + b_{kn}b'_{kn}) = -\det(B).
\end{aligned}$$

# Theorem 4.2.A(2) (continued)

**Proof (continued).** Since $n > 2$, we can choose a $k$th row for expansion by minors, where $k \notin \{r, i\}$. Consider the cofactors

$$(-1)^{k+j}|A_{kj}| \text{ and } (-1)^{k+j}|B_{kj}|.$$

These numbers must have opposite signs, by our induction hypothesis, since the minor matrices $A_{kj}$ and $B_{kj}$ have size $(n-1) \times (n-1)$, and $B_{kj}$ can be obtained from $A_{kj}$ by interchanging two rows (namely, the $i$th and $r$th rows). That is, $|B_{kj}| = -|A_{kj}|$ and so $b'_{kj} = -a'_{kj}$. So applying Theorem 4.2, "General Expansion by Minors," we can compute det$(B)$ by expanding along the $k$th row of $A$ and the $k$th row of $B$ we find:

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&= b_{k1}a'_{k1} + b_{k2}a'_{k2} + \cdots + b_{kn}a'_{kn} \\
&\quad \text{since the } k\text{th row of } A \text{ is the same as the } k\text{th row of } B \\
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&= -(b_{k1}b'_{k1} + b_{k2}b'_{k2} + \cdots + b_{kn}b'_{kn}) = -\det(B).
\end{aligned}$$

# Theorem 4.2.A(3), The Equal-Rows Property

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

3. The Equal-Rows Property: If two rows of a square matrix $A$ are equal, then $\det(A) = 0$.

**Proof (continued).** So the result holds for $k = n$. Therefore, by mathematical induction, (2) holds for all $n \times n$ matrices where $n$ is a natural number.

**(3)** Let $B$ be the matrix obtained from $A$ by interchanging the two equal rows (so $B = A$). By the Row-Interchange Property, $\det(B) = -\det(A)$. But since $B = A$, this implies $\det(B) = \det(A)$. Hence $\det(A) = -\det(A)$ and we must have $\det(A) = 0$.

# Theorem 4.2.A(3), The Equal-Rows Property

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

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**Proof (continued).** So the result holds for $k = n$. Therefore, by mathematical induction, (2) holds for all $n \times n$ matrices where $n$ is a natural number.

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# Theorem 4.2.A(4), The Scalar-Multiplication Property

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

4. The Scalar-Multiplication Property: If a single row of a square matrix $A$ is multiplied by a scalar $r$, the determinant of the resulting matrix if $r\det(A)$.

**Proof (continued). (4)** Let $r \in \mathbb{R}$ be a scalar and let $B$ be the matrix obtained from $A$ by multiplying the $k$th row of $A$ by $r$; so the $k$th row of $B$ is $[ra_{k1}, ra_{k2}, \ldots, ra_{kn}]$ so $b_{kj} = ra_{kj}$ for $j = 1, 2, \ldots, n$. Using Theorem 4.2, "General Expansion by Minors," we can compute $\det(A)$ by expanding along the $k$th row of $A$ to $\det(A)$ get in terms of cofactors that

$$\det(A) = a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn}.$$

# Theorem 4.2.A(4), The Scalar-Multiplication Property

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

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**Proof (continued). (4)** Let $r \in \mathbb{R}$ be a scalar and let $B$ be the matrix obtained from $A$ by multiplying the $k$th row of $A$ by $r$; so the $k$th row of $B$ is $[ra_{k1}, ra_{k2}, \ldots, ra_{kn}]$ so $b_{kj} = ra_{kj}$ for $j = 1, 2, \ldots, n$. Using Theorem 4.2, "General Expansion by Minors," we can compute $\det(A)$ by expanding along the $k$th row of $A$ to $\det(A)$ get in terms of cofactors that

$$\det(A) = a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn}.$$

Since all rows of $B$ equal the corresponding rows of $A$, except for the $k$th row, then the minors satisfy $A_{kj} = B_{kj}$ and the cofactors satisfy $a'_{kj} = b'_{kj}$ for $j = 1, 2, \ldots, n$.

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$$\det(A) = a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn}.$$

Since all rows of $B$ equal the corresponding rows of $A$, except for the $k$th row, then the minors satisfy $A_{kj} = B_{kj}$ and the cofactors satisfy $a'_{kj} = b'_{kj}$ for $j = 1, 2, \ldots, n$.

# Theorem 4.2.A(4), The Scalar-Multiplication Property (continued)

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

4. The Scalar-Multiplication Property: If a single row of a square matrix $A$ is multiplied by a scalar $r$, the determinant of the resulting matrix if $r\det(A)$.

**Proof (continued).** Finding $\det(B)$ by expanding along the $k$th row gives

$$\begin{aligned} \det(B) &= b_{k1}b'_{k1} + b_{k2}b'_{k2} + \cdots + b_{kn}b'_{kn} \\ &= ra_{k1}a'_{k1} + ra_{k2}a'_{k2} + \cdots + ra_{kn}a'_{kn} \\ &\quad \text{since } b_{kj} = ra_{kj} \text{ and } a'_{kj} = b'_{kj} \\ &= r(a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn}) = r\det(A). \end{aligned}$$

So the result holds for $k = n$. Therefore, by mathematical induction, (4) holds for all $n \times n$ matrices where $n$ is a natural number.

# Theorem 4.2.A(4), The Scalar-Multiplication Property (continued)

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

4. The Scalar-Multiplication Property: If a single row of a square matrix $A$ is multiplied by a scalar $r$, the determinant of the resulting matrix if $r\det(A)$.

**Proof (continued).** Finding $\det(B)$ by expanding along the $k$th row gives

$$\begin{aligned}\det(B) &= b_{k1}b'_{k1} + b_{k2}b'_{k2} + \cdots + b_{kn}b'_{kn} \\ &= ra_{k1}a'_{k1} + ra_{k2}a'_{k2} + \cdots + ra_{kn}a'_{kn} \\ &\qquad \text{since } b_{kj} = ra_{kj} \text{ and } a'_{kj} = b'_{kj} \\ &= r(a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn}) = r\det(A).\end{aligned}$$

So the result holds for $k = n$. Therefore, by mathematical induction, (4) holds for all $n \times n$ matrices where $n$ is a natural number.

# Theorem 4.2.A(5), The Row-Addition Property

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

5. The Row-Addition Property: If the product of one row of $A$ by a scalar $r$ is added to a different row of $A$, the determinant of the resulting matrix is the same as $\det(A)$.

**Proof (continued). (5)** The $i$th row of $A$ is $[a_{i1}, a_{i2}, \ldots, a_{in}]$ and the $k$th row of $A$ is $[a_{k1}, a_{k2}, \ldots, a_{kn}]$ where $i \neq k$. So if $B$ is obtained from $A$ by adding $r$ times Row $i$ to Row $k$, that is $[ra_{i1} + a_{k1}, ra_{i2} + a_{k2}, \ldots, ra_{in} + a_{kn}]$. As in the proof of Property 4, the minors satisfy $A_{kj} = B_{kj}$ and the cofactors satisfy $a'_{kj} = b'_{kj}$ for $j = 1, 2, \ldots, n$.

# Theorem 4.2.A(5), The Row-Addition Property

**Theorem 4.2.A. Properties of the Determinant.**
Let $A$ be a square matrix.

5. The Row-Addition Property: If the product of one row of $A$ by a scalar $r$ is added to a different row of $A$, the determinant of the resulting matrix is the same as $\det(A)$.

**Proof (continued). (5)** The $i$th row of $A$ is $[a_{i1}, a_{i2}, \ldots, a_{in}]$ and the $k$th row of $A$ is $[a_{k1}, a_{k2}, \ldots, a_{kn}]$ where $i \neq k$. So if $B$ is obtained from $A$ by adding $r$ times Row $i$ to Row $k$, that is $[ra_{i1} + a_{k1}, ra_{i2} + a_{k2}, \ldots, ra_{in} + a_{kn}]$. As in the proof of Property 4, the minors satisfy $A_{kj} = B_{kj}$ and the cofactors satisfy $a'_{kj} = b'_{kj}$ for $j = 1, 2, \ldots, n$.

# Theorem 4.2.A(5) The Row-Addition Property (continued)

**Proof (continued).** Using Theorem 4.2, "General Expansion by Minors," and expanding all determinant along the $k$th row, we have

$$\begin{aligned}
\det(B) &= b_{k1}b'_{k1} + b_{k2}b'_{k2} + \cdots + b_{kn}b'_{kn} \\
&= (ra_{i1} + a_{k1})a'_{k1} + (ra_{i2} + a_{k2})a'_{k2} + \cdots + (ra_{in} + a_{kn})a'_{kn} \\
&\quad \text{ since } b_{kj} = ra_{ij} + a_{kj} \text{ and } b'_{kj} = a_{kj} \\
&= r(a_{i1}a'_{k1} + a_{i2}a'_{k2} + \cdots + a_{in}a'_{kn}) \\
&\quad +(a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn}) \\
&= r\det(C) + \det(A)
\end{aligned}$$

where matrix $C$ is an $n \times n$ matrix with the same rows as matrix $A$, except that the $k$th row of $C$ is the same as the $i$th row of $A$. Since $i \neq k$, then Row $i$ and Row $k$ of $C$ are the same and so by Property 3, $\det(C) = 0$. Therefore, $\det(B) = \det(A)$, as claimed. □

# Theorem 4.2.A(5) The Row-Addition Property (continued)

**Proof (continued).** Using Theorem 4.2, "General Expansion by Minors," and expanding all determinant along the $k$th row, we have

$$\begin{aligned}
\det(B) &= b_{k1}b'_{k1} + b_{k2}b'_{k2} + \cdots + b_{kn}b'_{kn} \\
&= (ra_{i1} + a_{k1})a'_{k1} + (ra_{i2} + a_{k2})a'_{k2} + \cdots + (ra_{in} + a_{kn})a'_{kn} \\
&\quad \text{ since } b_{kj} = ra_{ij} + a_{kj} \text{ and } b'_{kj} = a_{kj} \\
&= r(a_{i1}a'_{k1} + a_{i2}a'_{k2} + \cdots + a_{in}a'_{kn}) \\
&\quad +(a_{k1}a'_{k1} + a_{k2}a'_{k2} + \cdots + a_{kn}a'_{kn}) \\
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where matrix $C$ is an $n \times n$ matrix with the same rows as matrix $A$, except that the $k$th row of $C$ is the same as the $i$th row of $A$. Since $i \neq k$, then Row $i$ and Row $k$ of $C$ are the same and so by Property 3, $\det(C) = 0$. Therefore, $\det(B) = \det(A)$, as claimed. □

# Page 261 Number 8

**Page 261 Number 8.** Use row reduction and Theorem 4.2.A to find

$$\det(A) \text{ for } A = \begin{bmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix}.$$

**Solution.** Row reducing we have

$$A = \begin{bmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix} \underset{R_4 \to R_4 - 2R_1}{\overset{R_2 \to R_2 - 3R_1}{\underset{\sim}{R_3 \to R_3 - 4R_1}}} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & -2 & 5 & -28 \\ 0 & 1 & 2 & -12 \end{bmatrix}$$

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$$\underset{R_4 \to R_4 - R_2}{\overset{R_3 \to R_3 + 2R_2}{\sim}} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & 11 & -62 \\ 0 & 0 & -1 & 5 \end{bmatrix} \overset{R_3 \leftrightarrow R_4}{\sim} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 11 & -62 \end{bmatrix}$$

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$$A = \begin{bmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix} \underset{R_4 \to R_4 - 2R_1}{\underbrace{\overset{R_2 \to R_2 - 3R_1}{R_3 \to R_3 - 4R_1}}} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & -2 & 5 & -28 \\ 0 & 1 & 2 & -12 \end{bmatrix}$$

$$\underset{R_4 \to R_4 - R_2}{\underbrace{R_3 \to R_3 + 2R_2}} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & 11 & -62 \\ 0 & 0 & -1 & 5 \end{bmatrix} \underbrace{R_3 \leftrightarrow R_4} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 11 & -62 \end{bmatrix}$$

# Page 261 Number 8 (continued)

**Page 261 Number 8.** Use row reduction and Theorem 4.2.A to find det$(A)$ for $A = \begin{bmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix}$.

**Solution (continued).** ...

$$\begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 11 & -62 \end{bmatrix} \underset{\sim}{\overset{R_4 \to R_4 + 11R_3}{}} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & -7 \end{bmatrix} = H.$$

So $A \sim H$ through a sequence of 6 row-additions and one row-interchange. Hence, by Theorem 4.2.A (Properties 2 and 5) $\det(A) = -\det(H)$. Now $H$ is upper-triangular, so as shown in Page 255 Example 4, $\det(H) = (2)(1)(-1)(-7) = 14$. Hence, $\boxed{\det(A) = -\det(H) = -14.}$ □

# Page 261 Number 8 (continued)

**Page 261 Number 8.** Use row reduction and Theorem 4.2.A to find $\det(A)$ for $A = \begin{bmatrix} 2 & 0 & -1 & 7 \\ 6 & 1 & 0 & 4 \\ 8 & -2 & 1 & 0 \\ 4 & 1 & 0 & 2 \end{bmatrix}$.

**Solution (continued).** ...

$$\begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 11 & -62 \end{bmatrix} \underset{\sim}{R_4 \to R_4 + 11R_3} \begin{bmatrix} 2 & 0 & -1 & 7 \\ 0 & 1 & 3 & -17 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & -7 \end{bmatrix} = H.$$

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# Theorem 4.3

**Theorem 4.3. Determinant Criterion for Invertibility.**
A square matrix $A$ is invertible if and only if $\det(A) \neq 0$. Equivalently, $A$ is singular if and only if $\det(A) = 0$.

**Proof.** As commented above, $A$ can be reduced to an echelon form $H$ without multiplying rows by scalars (i.e., "row scaling") so $\det(A) = \pm\det(H)$. The $H$ is upper triangular and so by Page 255 Example 4, the determinant of $A$ is the product of its diagonal entries.

# Theorem 4.3

**Theorem 4.3. Determinant Criterion for Invertibility.**
A square matrix $A$ is invertible if and only if $\det(A) \neq 0$. Equivalently, $A$ is singular if and only if $\det(A) = 0$.

**Proof.** As commented above, $A$ can be reduced to an echelon form $H$ without multiplying rows by scalars (i.e., "row scaling") so $\det(A) = \pm\det(H)$. The $H$ is upper triangular and so by Page 255 Example 4, the determinant of $A$ is the product of its diagonal entries. Now $A$ is invertible if and only if $A$ has only nonzero entries on its main diagonal since by Theorem 1.12, "Conditions for $A^{-1}$ to Exist," $A$ is invertible if and only if it is row equivalent to $\mathcal{I}$. So $\det(A) = \pm\det(H) \neq 0$ if and only if $A$ is invertible. □

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# Theorem 4.4

**Theorem 4.4. The Multiplicative Property.**
If $A$ and $B$ are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

**Proof.** First, if $A$ is a diagonal matrix then

$$AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

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$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1n} \\ a_{22}b_{21} & a_{22}b_{22} & \cdots & a_{22}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn}b_{n1} & a_{nn}b_{n2} & \cdots & a_{nn}b_{nn} \end{bmatrix}$$

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**Proof.** First, if $A$ is a diagonal matrix then

$$AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1n} \\ a_{22}b_{21} & a_{22}b_{22} & \cdots & a_{22}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn}b_{n1} & a_{nn}b_{n2} & \cdots & a_{nn}b_{nn} \end{bmatrix}$$

and so by Theorem 4.2.A(4), "The Scalar-Multiplication Property," $\det(AB) = a_{11}a_{22}\cdots a_{nn}\det(B) = \det(A)\det(B)$ because $A$ upper triangular implies $\det(A) = a_{11}a_{22}\cdots a_{nn}$ by Page 255 Example 4.

# Theorem 4.4

**Theorem 4.4. The Multiplicative Property.**
If $A$ and $B$ are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

**Proof.** First, if $A$ is a diagonal matrix then

$$AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

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# Theorem 4.4 (continued 1)

**Proof (continued).** Second, if $A$ is not invertible then $AB$ is not invertible by Exercise 30, so by Theorem 4.1, "Determinant Criterion for Invertibility," $\det(A) = \det(AB) = 0$.

Third, for $A$ invertible then as seen in the proof of Theorem 4.3, "Determinant Criterion for Invertibility," $A$ can be row reduced through row-interchange and row-addition elementary row operations to an upper-triangular matrix with nonzero entries on the diagonal. We can then use row-interchange and row-addition, as we would in the Gauss-Jordan Method, to reduce $A$ to a diagonal matrix where no diagonal entries are 0. So there is a matrix $E$, a product of elementary matrices corresponding to row-interchange and row-addition, such that $D = EA$ where $D$ is the diagonal matrix just described. Then Theorem 4.2.A(2) and (5), "Row-Interchange Property" and "Row-Addition Property," imply that $\det(A) = (-1)^r \det(D)$ where $r$ is the number of row interchanges used in the row reduction of $A$ to $D$.

# Theorem 4.4 (continued 1)

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# Theorem 4.4 (continued 2)

**Theorem 4.4. The Multiplicative Property.**
If $A$ and $B$ are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

**Proof (continued).** The same sequence of elementary row operations will reduce the matrix $AB$ to the matrix $E(AB) = (EA)B = DB$. So, similar to the determinant of $A$, we have $\det(AB) = (-1)^r \det(DB)$. Therefore,

$$\begin{aligned} \det(AB) &= (-1)^r \det(DB) \\ &= (-1)^r \det(D)\det(B) \text{ since we showed that} \\ &\qquad \text{the theorem holds if the first matrix is diagonal} \\ &= ((-1)^r \det(D))\det(B) \\ &= \det(A)\det(B) \text{ since } \det(A) = (-1)^r \det(D). \end{aligned}$$

Hence, $\det(AB) = \det(A)\det(B)$ in general. □

# Page 262 Number 28

**Page 262 Number 28.** Find the values of $\lambda$ for which the matrix
$A = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 4 \\ 0 & 1 & 1-\lambda \end{bmatrix}$ is singular.

**Solution.** By Theorem 4.3, "Determinant Criterion for Invertibility," $A$ is singular if and only if $\det(A) = 0$. We have by the definition of determinant of a $3 \times 3$ matrix that

$$\begin{aligned} \det(A) &= (2-\lambda)\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} - (0) + (0) \\ &= (2-\lambda)\left((1-\lambda)(1-\lambda) - (4)(1)\right) = (2-\lambda)(1 - 2\lambda + \lambda^2 - 4) \\ &= (2-\lambda)(\lambda^2 - 2\lambda - 3) = (2-\lambda)(\lambda-3)(\lambda+1). \end{aligned}$$

So $\det(A) = 0$ if and only if $\lambda = -1$, $\lambda = 2$, or $\lambda = 3$. That is, $A$ is singular if and only if $\boxed{\lambda = -1, \lambda = 2, \text{ or } \lambda = 3.}$

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**Page 262 Number 30.** If $A$ and $B$ are $n \times n$ matrices and if $A$ is singular, prove (without using Theorem 4.4) that $AB$ is also singular.

**Solution.** We give a proof by contradiction. Let $A$ be singular and ASSUME that $AB$ is nonsingular. Then there is $(AB)^{-1}$ where $(AB)(AB)^{-1} = \mathcal{I}$.

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# Page 262 Number 32

**Page 262 Number 32.** If $A$ and $C$ are $n \times n$ matrices with $C$ invertible, prove that $\det(A) = \det(C^{-1}AC)$. HINT: By Exercise 31, for invertible $C$ we have $\det(C^{-1}) = 1/\det(C)$.

**Prove.** By Theorem 4.4, "The Multiplicative Property,"

$$\det(C^{-1}AC) = \det(C^{-1}(AC)) = \det(C^{-1})\det(AC) = \det(C^{-1})\det(A)\det(C)$$

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$$\det(C^{-1}AC) = (1/\det(C))\det(A)\det(C) = \det(A),$$

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