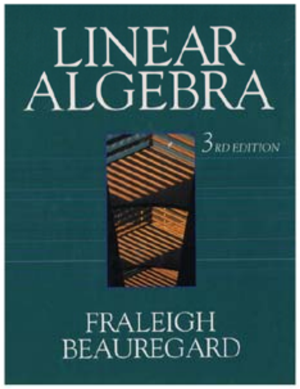
Linear Algebra

Chapter 4: Determinants

Section 4.3. Computation of Determinants and Cramer's Rule—Proofs of Theorems



Page 271 Number 6

Page 271 Number 6. Find $\det(A)$ where $A = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ -1 & 4 & 1 & 0 & 0 \\ 0 & -3 & 5 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$.

Solution. We state the elementary row operations and keep track of how they affect the determinant based on Theorem 4.2.A, "Properties of the Determinant." We have:

$$\det(A) = \begin{vmatrix} 3 & 2 & 0 & 0 & 0 \\ -1 & 4 & 1 & 0 & 0 \\ 0 & -3 & 5 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 & 2 \end{vmatrix}$$

Page 271 Number 6 (continued 1)

Solution (continued).

$$\begin{vmatrix} 3 & 2 & 0 & 0 & 0 \\ -1 & 4 & 1 & 0 & 0 \\ 0 & -3 & 5 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 & 2 \end{vmatrix} = - \begin{vmatrix} -1 & 4 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & -3 & 5 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 & 2 \end{vmatrix} \quad \begin{matrix}\text{Row Exchange:} \\ R_1 \leftrightarrow R_2\end{matrix}$$

$$= - \begin{vmatrix} -1 & 4 & 1 & 0 & 0 \\ 0 & 14 & 3 & 0 & 0 \\ 0 & -3 & 5 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 6 \end{vmatrix} \quad \begin{matrix}\text{Row Addition:} \\ R_2 \to R_2 + 3R_1 \text{ and } R_5 \to R_5 + R_4\end{matrix}$$

Page 271 Number 6 (continued 2)

Solution (continued). ...

$$- \begin{vmatrix} -1 & 4 & 1 & 0 & 0 \\ 0 & 14 & 3 & 0 & 0 \\ 0 & -3 & 5 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 6 \end{vmatrix} = - \begin{vmatrix} -1 & 4 & 1 & 0 & 0 \\ 0 & 14 & 3 & 0 & 0 \\ 0 & 0 & 79/14 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 6 \end{vmatrix} \quad \begin{matrix}\text{Row Addition:} \\ R_3 \to R_3 + (3/14)R_2\end{matrix}$$

$$= -(-1)(14)(79/14)(1)(6) = \boxed{474} \text{ by Example 4.2.4.}$$

□

Theorem 4.5

Theorem 4.5. Cramer's Rule.
Consider the linear system $A\vec{x} = \vec{b}$, where $A = [a_{ij}]$ is an $n \times n$ invertible matrix,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

The system has a unique solution given by

$$x_k = \frac{\det(B_k)}{\det(A)} \text{ for } k = 1, 2, \ldots, n,$$

where B_k is the matrix obtained from A by replacing the kth column vector of A by the column vector $\vec{b}$.

Proof. Since A is invertible, we know that the linear system $A\vec{x} = \vec{b}$ has a unique solution by Theorem 1.16. Let $\vec{x}$ be this unique solution.

Theorem 4.5 (continued 1)

Proof (continued). Let X_k be the matrix obtained from the $n \times n$ identity matrix by replacing its kth column vector by the column vector $\vec{x}$, so

$$X_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & x_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & x_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & x_3 & 0 & 0 & \cdots & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & x_k & 0 & 0 & \cdots & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & x_n & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

We now compute the product AX_k. If $j \neq k$, then the jth column of AX_k is the product of A and the jth column of the identity matrix, which is just the jth column of A. If $j = k$, then the jth column of AX_k is $A\vec{x} = \vec{b}$. Thus AX_k is the matrix obtained from A by replacing the kth column of A by the column vector $\vec{b}$.

Theorem 4.5 (continued 2)

Proof (continued). That is, AX_k is the matrix B_k described in the statement of the theorem. From the equation $AX_k = B_k$ and Theorem 4.4, "The Multiplicative Property," we have

$$\det(A)\det(X_k) = \det(B_k).$$

Computing $\det(X_k)$ by expanding by minors across the kth row (applying Theorem 4.2, "General Expansion by Minors"), we see that $\det(X_k) = x_k$ and thus $\det(A)x_k = \det(B_k)$. Because A is invertible, we know that $\det(A) \neq 0$ by Theorem 4.3, "Determinant Criterion for Invertibility," and so $x_k = \det(B_k)/\det(A)$ as claimed. □

Page 272 Number 26

Page 272 Number 26. Use Cramer's Rule to solve $\begin{array}{rcrcl} 3x_1 & + & x_2 & = & 5 \\ 2x_1 & + & x_2 & = & 0 \end{array}$.

Solution. We have $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. So $B_1 = \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 3 & 5 \\ 2 & 0 \end{bmatrix}$. Next, $\det(A) = (3)(1) - (1)(2) = 1$, $\det(B_1) = (5)(1) - (1)(0) = 5$, and $\det(B_2) = (3)(0) - (5)(2) = -10$. So by Cramer's Rule,

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{5}{1} = 5 \text{ and } x_2 = \frac{\det(B_2)}{\det(A)} = \frac{-10}{1} = -10.$$

So $\boxed{x_1 = 5 \text{ and } x_2 = -10.}$ □

Page 272 Number 18

Page 272 Number 18. Find the adjoint of $A = \begin{bmatrix} 3 & 0 & 3 \\ 4 & 1 & -2 \\ -5 & 1 & 4 \end{bmatrix}$.

Solution. First, we compute the 9 cofactors:

$$a'_{11} = \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = 6,\ a'_{12} = -\begin{vmatrix} 4 & -2 \\ -5 & 4 \end{vmatrix} = -6,\ a'_{13} = \begin{vmatrix} 4 & 1 \\ -5 & 1 \end{vmatrix} = 9,$$

$$a'_{21} = -\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} = 3,\ a'_{22} = \begin{vmatrix} 3 & 3 \\ -5 & 4 \end{vmatrix} = 27,\ a'_{23} = -\begin{vmatrix} 3 & 0 \\ -5 & 1 \end{vmatrix} = -3,$$

$$a'_{31} = \begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix} = -3,\ a'_{32} = -\begin{vmatrix} 3 & 3 \\ 4 & -2 \end{vmatrix} = 18,\ a'_{33} = \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} = 3,$$

so $A' = [a'_{ij}] = \begin{bmatrix} 6 & -6 & 9 \\ 3 & 27 & -3 \\ -3 & 18 & 3 \end{bmatrix} \ldots$

Page 272 Number 18 (continued)

Page 272 Number 18. Find the adjoint of $A = \begin{bmatrix} 3 & 0 & 3 \\ 4 & 1 & -2 \\ -5 & 1 & 4 \end{bmatrix}$.

Solution (continued). $\ldots A' = [a'_{ij}] = \begin{bmatrix} 6 & -6 & 9 \\ 3 & 27 & -3 \\ -3 & 18 & 3 \end{bmatrix}$ and

$$\mathrm{adj}(A) = (A')^T = \boxed{\begin{bmatrix} 6 & 3 & -3 \\ -6 & 27 & 18 \\ 9 & -3 & 3 \end{bmatrix}}.$$

□

Theorem 4.6

Theorem 4.6. Property of the Adjoint.
Let A be $n \times n$. Then

$$(\mathrm{adj}(A))A = A\,\mathrm{adj}(A) = (\det(A))\mathcal{I}.$$

Proof. Let $A = [a_{ij}]$. Define B as the matrix which results from replacing Row j of A with Row i of A. Then, by Theorem 4.2.A, "Properties of Determinants,"

$$\det(B) = \begin{cases} \det(A) & \text{if } i = j \text{ (since } B = A) \\ 0 & \text{if } i \neq j, \text{ by Theorem 4.2.A(3), "Equal Row Property."} \end{cases}$$

Now we can expand $\det(B)$ about the jth row of B to get by Theorem 4.2, "General Expansion by Minors," that $\det(B) = \sum_{s=1}^n a_{is}a'_{js}$ and so

$$\sum_{s=1}^n a_{is}a'_{js} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \qquad (2)$$

Notice that the (i,j) entry of $A(A')^T$ is $\sum_{k=1}^n a_{ik}a'_{jk}$ where $A' = [a'_{ij}]$.

Theorem 4.6 (continued)

Theorem 4.6. Property of the Adjoint.
Let A be $n \times n$. Then

$$(\mathrm{adj}(A))A = A\,\mathrm{adj}(A) = (\det(A))\mathcal{I}.$$

Proof (continued). Since we can express the right-hand side of (2) as $\det(A)\mathcal{I}$, then we have $A(A')^T = A\,\mathrm{adj}(A) = \det(A)\mathcal{I}$.

Similarly if matrix C results from replacing Column i of A with Column j of A and by computing $\det(C)$ by expanding along the ith column of C we get

$$\sum_{r=1}^n a'_{ri}a'_{rj} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

and so $(A')^T A = \mathrm{adj}(A)A = \det(A)\mathcal{I}$. Hence, $\mathrm{adj}(A)A = A\,\mathrm{adj}(A) = \det(A)\mathcal{I}$, as claimed. □

Page 272 Number 18

Page 272 Number 18. Find the inverse of $A = \begin{bmatrix} 3 & 0 & 3 \\ 4 & 1 & -2 \\ -5 & 1 & 4 \end{bmatrix}$ using $\mathrm{adj}(A)$.

Solution. First, we compute $\det(A)$ by expanding along the first row:

$$\det(A) = (3)\begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} - (0) + (3)\begin{vmatrix} 4 & 1 \\ -5 & 1 \end{vmatrix} = 3(6) + 3(9) = 45.$$

So by Corollary 4.3.A, "Formula for A^{-1}," we have (using $\mathrm{adj}(A)$ computed above)

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)} = \frac{1}{45}\begin{bmatrix} 6 & 3 & -3 \\ -6 & 27 & 18 \\ 9 & -3 & 3 \end{bmatrix} = \boxed{\frac{1}{15}\begin{bmatrix} 2 & 1 & -1 \\ -2 & 9 & 6 \\ 3 & -1 & 1 \end{bmatrix}}.$$

□

Page 272 Number 22

Page 272 Number 22. Given that $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A^{-1}) = 3$, find the matrix A.

Solution. We know from Corollary 4.3.A, "Formula for A^{-1}," that $A^{-1} = \mathrm{adj}(A)/\det(A)$. Now $\det(A^{-1}) = 1/\det(A)$ by Exercise 4.2.31, so $\det(A) = 1/\det(A^{-1}) = 1/3$. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $a'_{11} = a_{22}$, $a'_{12} = -a_{21}$, $a'_{21} = -a_{12}$, and $a'_{22} = a_{11}$. So $A' = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$ and $\mathrm{adj}(A) = (A')^T = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \det(A)A^{-1} = \frac{1}{3}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and so $a_{11} = d/3$, $a_{12} = -b/3$, $a_{21} = -c/3$, and $a_{22} = a/3$. Therefore

$$\boxed{A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} d/3 & -b/3 \\ -c/3 & a/3 \end{bmatrix}}.$$ □

Page 273 Number 36

Page 273 Number 36. Prove that the inverse of a nonsingular upper-triangular matrix is upper triangular.

Solution. Let $A = [a_{ij}]$ be a (square) nonsingular upper triangular matrix; that is, $a_{ij} = 0$ for $i > j$. Now the minor matrix A_{ij} (obtained from A by eliminating Row i and Column j from A) is upper triangular for $i < j$:

$$\begin{array}{cc}
 & \begin{array}{ccccccccccccc} &&&&&&&&& \text{Column } j &&& \end{array} \\
\begin{array}{c} \\ \\ \\ \\ \\ \text{Row } i \\ \\ \\ \\ \\ \\ \\ \\ \end{array} &
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1,i-1} & a_{1i} & a_{1,i+1} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\
0 & a_{22} & a_{23} & \cdots & a_{2,i-1} & a_{2i} & a_{2,i+1} & \cdots & a_{2,j-1} & a_{2j} & a_{2,j+1} & \cdots & a_{2n} \\
0 & 0 & a_{33} & \cdots & a_{3,i-1} & a_{3i} & a_{3,i+1} & \cdots & a_{3,j-1} & a_{3j} & a_{3,j+1} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{i-1,i-1} & a_{i-1,i} & a_{i-1,i+1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\
0 & 0 & 0 & \cdots & 0 & a_{ii} & a_{i,i+1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\
0 & 0 & 0 & \cdots & 0 & 0 & a_{i+1,i+1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{j-1,j-1} & a_{j-1,j} & a_{j-1,j+1} & \cdots & a_{j-1,n} \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{jj} & a_{j,j+1} & \cdots & a_{jn} \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & a_{j+1,j+1} & \cdots & a_{j+1,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{nn}
\end{bmatrix}
\end{array}$$

Page 273 Number 36 (continued)

Page 273 Number 36. Prove that the inverse of a nonsingular upper-triangular matrix is upper triangular.

Solution (continued). Then, with $i < j$, $(n-1) \times (n-1)$ minor matrix A_{ij} has a 0 in *its* (i,i) entry (it is element $a_{i+1,i} = 0$ in matrix A). So for $i < j$, A_{ij} is upper triangular with a 0 on the diagonal. By Example 4.2.4 (the determinant of an upper triangular square matrix is the product of the diagonal entries), $\det(A_{ij}) = 0$ and so cofactor $a_{ij} = (-1)^{i+j}\det(A_{ij}) = 0$ for $i < j$. So matrix A' has 0 in entry (i,j) whenever $i < j$. That is, A' is lower triangular. Hence $\mathrm{adj}(A) = (A')^T$ is upper triangular. Since A is nonsingular then by Theorem 4.3, "Determinant Criterion for Invertibility," $\det(A) \neq 0$. By Corollary 4.3.A, "A Formula for the Inverse of an Invertible Matrix," $A^{-1} = \dfrac{1}{\det(A)}\mathrm{adj}(A)$ and so A^{-1} is also upper triangular. □

Page 273 Number 38

Page 273 Number 38. Let A be an $n \times n$ nonsingular matrix. Prove that $\det(\text{adj}(A)) = \det(A)^{n-1}$.

Solution. By Corollary 4.3.A, "A Formula for A^{-1}," $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$. By Exercise 4.2.31, $\det(A^{-1}) = 1/\det(A)$, so we have

$$\begin{aligned} \frac{1}{\det(A)} &= \det(A^{-1}) = \det\left(\frac{1}{\det(A)}\text{adj}(A)\right) \\ &= \frac{1}{\det(A)^n}\det(\text{adj}(A)) \text{ by Theorem 4.2.A(4), "Scalar Multiplication Property," applied to each of the } n \text{ rows of adj}(A). \end{aligned}$$

So $\det(\text{adj}(A)) = \det(A)^n/\det(A) = \det(A)^{n-1}$, as claimed. □

...

Page 273 Number 38

Page 273 Number 38 (continued)

Page 273 Number 38. Let A be an $n \times n$ nonsingular matrix. Prove that $\det(\text{adj}(A)) = \det(A)^{n-1}$.

Note. This result also holds if A is an $n \times n$ singular matrix. If A is singular then $\det(A) = 0$ by Theorem 4.3, "Determinant Criterion for Invertibility." By Exercise 37, A is invertible if and only if $\text{adj}(A)$ is invertible. So $\det(A) = 0$ implies $\det(\text{adj}(A)) = 0$ (again, by Theorem 4.3), and so Exercise 38 holds for nonsingular square matrices as well.