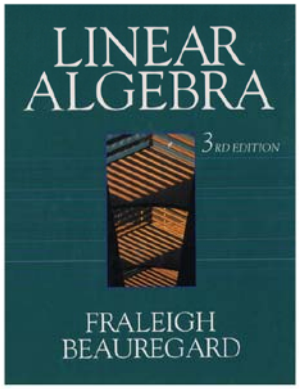
Linear Algebra

Chapter 4: Determinants

Section 4.4. Linear Transformations and Determinants—Proofs of Theorems



Theorem B.2

Theorem B.2. Property of $\det(A^TA)$**.**
Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \in \mathbb{R}^m$ and let A be the $m \times n$ matrix with jth column $\vec{a}_j$. Let B be the $m \times n$ matrix obtained from A by replacing the first column of A by the vector $\vec{b} = \vec{a}_1 - r_2\vec{a}_2 - r_3\vec{a}_3 - \cdots - r_n\vec{a}_n$ for scalars $r_2, r_3, \ldots, r_n$. Then $\det(A^TA) = \det(B^TB)$.

Proof. Matrix B can be obtained from matrix A by a sequence of $n-1$ elementary *column*-addition operations. Each of the elementary column operations can be performed on A by multiplying A on the *right* by an elementary matrix formed by exerting the same elementary *column*-addition on the identity matrix $\mathcal{I}$ by Exercises 1.5.36 and 1.5.37. Each such elementary matrix has the same determinant as the identity matrix (namely, 1) by Theorem 4.2.A(1) and (5). Let E by the product of these elementary matrices so that $B = AE$. By Theorem 4.4, "The Multiplicative Property," $\det(E) = 1$.

Theorem B.2 (continued)

Theorem B.2. Property of $\det(A^TA)$**.**
Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \in \mathbb{R}^m$ and let A be the $m \times n$ matrix with jth column $\vec{a}_j$. Let B be the $m \times n$ matrix obtained from A by replacing the first column of A by the vector $\vec{b} = \vec{a}_1 - r_2\vec{a}_2 - r_3\vec{a}_3 - \cdots - r_n\vec{a}_n$ for scalars $r_2, r_3, \ldots, r_n$. Then $\det(A^TA) = \det(B^TB)$.

Proof (continued). Then

$$\begin{aligned}\det(B^TB) &= \det((AE)^T(AE)) = \det(E^TA^TAE) = \det E^T(A^TA)E) \\ &= \det(E^T)\det(A^TA)\det(E) \text{ by Theorem 4.4} \\ &= 1\det(A^TA)a = \det(A^TA),\end{aligned}$$

as claimed. □

Theorem 4.7

Theorem 4.7. Volume of an n**-Box.**
The volume of the n-box in $\mathbb{R}^m$ determined by independent vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is given by (Volume) $= \sqrt{\det(A^TA)}$ where A is the $m \times n$ matrix with $\vec{a}_j$ as its jth column vector.

Proof. We give a proof based on mathematical induction. As argued above, the result holds for $n = 1$ and $n = 2$ (and for $n = 3$ if we interchange $\vec{a}_1$ and $\vec{a}_3$). Let $n > 2$ and assume (the induction hypothesis) that the claim holds for all k-boxes where $1 \le k \le n-1$. With $\vec{b} = \vec{a}_1 - \vec{p} = \vec{a}_1 - \text{proj}_{\text{sp}(\vec{a}_2,\vec{a}_3,\ldots,\vec{a}_n)}(\vec{a}_1)$ then $\vec{p} \in \text{sp}(\vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n)$ so that $\vec{p} = r_2\vec{a}_2 + r_3\vec{a}_3 + \cdots + r_n\vec{a}_n$ for some scalars $r_2, r_3, \ldots, r_n$, so $\vec{b} = \vec{a}_1 - \vec{p} = \vec{a}_1 - r_2\vec{a}_2 - r_3\vec{a}_3 - \cdots - r_n\vec{a}_n$. Let B be the matrix obtained from A by replacing the first *column* vector $\vec{a}_1$ of A by the vector $\vec{b}$ (as in Theorem B.2). Now $\vec{b} \in W^\perp$ where $W = \text{sp}(\vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n)$, so $\vec{b} \cdot \vec{a}_i = 0$ for $i = 2, 3, \ldots, n$ and ...

Theorem 4.7 (continued 1)

Proof (continued).

$$B^TB = \begin{bmatrix} \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{a}_2 & \vec{b}\cdot\vec{a}_3 & \cdots & \vec{b}\cdot\vec{a}_n \\ \vec{a}_2\cdot\vec{b} & \vec{a}_2\cdot\vec{a}_2 & \vec{a}_2\cdot\vec{a}_3 & \cdots & \vec{a}_2\cdot\vec{a}_n \\ \vec{a}_3\cdot\vec{b} & \vec{a}_3\cdot\vec{a}_2 & \vec{a}_3\cdot\vec{a}_3 & \cdots & \vec{a}_3\cdot\vec{a}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{a}_n\cdot\vec{b} & \vec{a}_n\cdot\vec{a}_2 & \vec{a}_n\cdot\vec{a}_3 & \cdots & \vec{a}_n\cdot\vec{a}_n \end{bmatrix} = \begin{bmatrix} \vec{b}\cdot\vec{b} & 0 & 0 & \cdots & 0 \\ 0 & \vec{a}_2\cdot\vec{a}_2 & \vec{a}_2\cdot\vec{a}_3 & \cdots & \vec{a}_2\cdot\vec{a}_n \\ 0 & \vec{a}_3\cdot\vec{a}_2 & \vec{a}_3\cdot\vec{a}_3 & \cdots & \vec{a}_3\cdot\vec{a}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vec{a}_n\cdot\vec{a}_2 & \vec{a}_n\cdot\vec{a}_3 & \cdots & \vec{a}_n\cdot\vec{a}_n \end{bmatrix}.$$

So expanding along the first row we have . . .

Theorem 4.7 (continued 2)

Proof (continued).

$$\det(B^TB) = \|\vec{b}\|^2 \begin{bmatrix} \vec{a}_2\cdot\vec{a}_2 & \vec{a}_2\cdot\vec{a}_3 & \cdots & \vec{a}_2\cdot\vec{a}_n \\ \vec{a}_3\cdot\vec{a}_2 & \vec{a}_3\cdot\vec{a}_3 & \cdots & \vec{a}_3\cdot\vec{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_n\cdot\vec{a}_2 & \vec{a}_n\cdot\vec{a}_3 & \cdots & \vec{a}_n\cdot\vec{a}_n \end{bmatrix}.$$

By the induction hypothesis, the square of the volume of the base of the n-box is the $(n-1)$-box determined by the $(n-1)$ vectors $\vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ and so

$$\begin{aligned} \det(B^TB) &= \|\vec{b}\|^2(\text{Volume of the base})^2 \\ &= (\text{Volume of } n\text{-box})^2 \text{ by Definition B.1} \\ &= \det(A^TA) \text{ by Theorem B.2.} \end{aligned}$$

So the claim holds for $k = n$ and by induction holds for all $n \in \mathbb{N}$, as claimed. □

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Page 284 Number 4. Find the volume of the 4-box in $\mathbb{R}^5$ determined by the vectors $[1,1,1,0,1]$, $[0,1,1,0,0]$, $[3,0,1,0,0]$, and $[1,-1,0,0,1]$.

Solution. We let

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Then, by Theorem 4.7,

$$(\text{Volume}) = \sqrt{\det(A^TA)} = \begin{vmatrix} 4 & 2 & 4 & 1 \\ 2 & 2 & 1 & -1 \\ 4 & 1 & 10 & 3 \\ 1 & -1 & 3 & 3 \end{vmatrix}^{1/2}$$

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Solution (continued). . . .

$$(\text{Volume}) = \begin{vmatrix} 0 & 6 & -8 & -11 \\ 0 & 4 & -5 & -7 \\ 0 & 5 & -2 & -9 \\ 1 & -1 & 3 & 3 \end{vmatrix} \quad \begin{array}{c} \text{using the row operations} \\ R_1 \to R_1 - 4R_4, \\ R_2 \to R_2 - 2R_2, \\ R_3 \to R_3 - 4R_4 \end{array}$$

$$= -\begin{vmatrix} 6 & -8 & -11 \\ 4 & -5 & -7 \\ 5 & -2 & -9 \end{vmatrix}$$

$$= -\left(6\begin{vmatrix} -5 & -7 \\ -2 & -9 \end{vmatrix} - (-8)\begin{vmatrix} 4 & -7 \\ 5 & -9 \end{vmatrix} + (-11)\begin{vmatrix} 4 & -5 \\ 5 & -2 \end{vmatrix}\right)$$

$$= -6(31) - 8(-1) + 11(17(= -186 + 8 + 187 = \boxed{9}.$$

□

Corollary

Corollary. Independence of Order.
The volume of a box determined by the independent vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ (as defined in Definition B.1) is independent of the order of the vectors.

Proof. A rearrangement of the sequence $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ of vectors corresponds to the same rearrangement of the *columns* of matrix A. This rearrangement of columns can be performed on A by multiplying A on the *right* by a sequence of elementary matrices formed by interchanging two columns of an identity matrix, by Exercises 1.5.36 and 1.5.37. Each such elementary matrix has a determinant of -1 times the determinant of the identity matrix (namely, 1) by Theorem 4.2.A(1) and (2). Let E be the product of these elementary matrices so that $B = AE$ where B has the same columns as A, only rearranged. Then $\det(E) = \pm 1$ and so $\det(E^T)\det(E) = (\pm 1)^2 = 1$.

Corollary (continued)

Corollary. Independence of Order.
The volume of a box determined by the independent vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ (as defined in Definition B.1) is independent of the order of the vectors.

Proof (continued). Then

$$\begin{aligned} \det(B^T B) &= \det((AE)^T(AE)) = \det(E^T A^T A E) \\ &= \det(E^T)\det(A^T A)\det(E) \text{ by Theorem 4.4} \\ &= \det(A^T A) = (\text{Volume of the } n\text{-box}) \text{ by Theorem 4.7.} \end{aligned}$$

Since B has the same columns as A, but in an arbitrary order, the result follows. □

Corollary

Corollary. Volume of an n-Box in $\mathbb{R}^n$.
If A is an $n \times n$ matrix with independent column vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ then $|\det(A)|$ is the volume of the n-box in $\mathbb{R}^n$ determined by these n vectors.

Proof. We have

$$\begin{aligned} (\text{Volume of } n\text{-box}) &= \sqrt{\det(A^T A)} \text{ by Theorem 4.7} \\ &= \sqrt{\det(A^T)\det(A)} \text{ by Theorem 4.4} \\ &= \sqrt{\det(A)\det(A)} \text{ by Theorem 4.2.A(1)} \\ &= |\det(A)|. \end{aligned}$$

□

Page 284 Number 8.

Page 284 Number 8. Find the volume of the 3-box determined by $[-1, 4, 7]$, $[3, -2, -1]$, and $[4, 0, 2]$ in $\mathbb{R}^3$.

Solution. Applying the second Corollary to Theorem 4.7, we let $A = \begin{bmatrix} -1 & 3 & 4 \\ 4 & -2 & 0 \\ 7 & -1 & 2 \end{bmatrix}$ and then we have

$$\begin{aligned} (\text{Volume of 3-box}) &= |\det(A)| = \left| \begin{vmatrix} -1 & 3 & 4 \\ 4 & -2 & 0 \\ 7 & -1 & 2 \end{vmatrix} \right| \\ &= \left| (-1)\begin{vmatrix} -2 & 0 \\ -1 & 2 \end{vmatrix} - (3)\begin{vmatrix} 4 & 0 \\ 7 & 2 \end{vmatrix} + (4)\begin{vmatrix} 4 & -2 \\ 7 & -1 \end{vmatrix} \right| \\ &= |(-1)(-4) - (3)(8) + (4)(10)| = \boxed{20.} \end{aligned}$$

□

Page 284 Number 22.

Page 284 Number 22. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T([x, y, z]) = [x - 2y, 3x + z, 4x + 3y]$. Find the volume of the image of the 3-box $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ under T.

Solution. The 3-box is determined by $\vec{b}_1 = [1, 0, 0]$, $\vec{b}_2 = [0, 1, 0]$, $\vec{b}_3 = [0, 0, 1]$ so we take $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Next, $T(\hat{\imath}) = T([1, 0, 0]) = [1, 3, 4]$, $T(\hat{\jmath}) = T([0, 1, 0]) = [-2, 0, 3]$, and $T(\hat{k}) = T([0, 0, 1]) = [0, 1, 0]$ so the standard matrix representation of T is $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & 1 \\ 4 & 3 & 0 \end{bmatrix}$. Now $\det(A) = 0 - (-1)\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix} + 0 = -(1)(11) = -11$. Since the volume of the 3-cube is 1, then the volume of the image is $|\det(A)\det(B)| = \boxed{11.}$ □

Page 284 Number 32.

Page 284 Number 32. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by $T([x, y, z]) = [x - y, x, -y, 2x + y]$. Find the area of the image under T of the region $x^2 + y^2 \le 0$ in $\mathbb{R}^2$.

Solution. The area of $x^2 + y^2 \le 9$ in $\mathbb{R}^2$ is $V = 9\pi$. The standard matrix representation of T is $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$ and so $A^T = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 0 & -1 & 1 \end{bmatrix}$. Now $A^TA = \begin{bmatrix} 6 & 1 \\ 1 & 3 \end{bmatrix}$ and $\det(A^TA) = 17$. So the volume of the image under T of the region is $\sqrt{\det(A^TA)}V = \sqrt{17}9\pi = \boxed{9\pi\sqrt{17}.}$ □