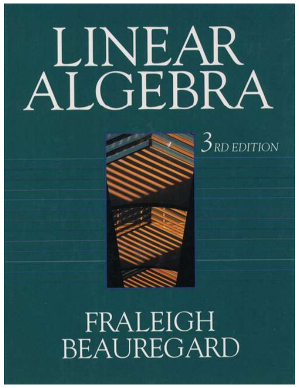
Linear Algebra

Chapter 5: Eigenvalues and Eigenvectors

Section 5.1. Eigenvalues and Eigenvectors—Proofs of Theorems



Page 300 Number 8

Page 300 Number 8. Find the characteristic polynomial, the real eigenvalues, and the corresponding eigenvectors for $A = \begin{bmatrix} -1 & 0 & 0 \\ -4 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix}$.

Solution. We have

$$A - \lambda\mathcal{I} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1-\lambda & 0 & 0 \\ -4 & 2-\lambda & -1 \\ 4 & 0 & 3-\lambda \end{bmatrix}.$$

So the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda\mathcal{I}) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ -4 & 2-\lambda & -1 \\ 4 & 0 & 3-\lambda \end{vmatrix} \\ &= (-1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 0 & 3-\lambda \end{vmatrix} - (0) + (0) \\ &= (-1-\lambda)\left((2-\lambda)(3-\lambda) - (-1)(0)\right) = \boxed{(-1-\lambda)(2-\lambda)(3-\lambda).} \end{aligned}$$

Page 300 Number 8 (continued 1)

Solution (continued). So $\boxed{\text{the eigenvalues are } \lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 3.}$ To find the eigenvectors corresponding to each eigenvalue, we consider the formula $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$ (see Note 5.1.A):

$\underline{\lambda_1 = -1.}$ With $\vec{v}_1 = [v_1, v_2, v_3]^T$ an eigenvector corresponding to the eigenvalue $\lambda_1 = -1$ we need $(A - \lambda_1\mathcal{I})\vec{v}_1 = \vec{0}$. So we consider the augmented matrix

$$\left[\begin{array}{ccc|c} -1-(-1) & 0 & 0 & 0 \\ -4 & 2-(-1) & -1 & 0 \\ 4 & 0 & 3-(-1) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -4 & 3 & -1 & 0 \\ 4 & 0 & 4 & 0 \end{array}\right]$$

$$\underset{\sim}{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -4 & 3 & -1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \underset{\sim}{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -4 & 0 & -4 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3/3}{\underset{\sim}{R_2 \to R_2/(-4)}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Page 300 Number 8 (continued 2)

Solution (continued).

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \underset{\sim}{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{rcrcr} v_1 & & + & v_3 & = & 0 \\ & v_2 & + & v_3 & = & 0 \\ & & & 0 & = & 0 \end{array}$, or $\begin{array}{rcr} v_1 & = & -v_3 \\ v_2 & = & -v_3 \\ v_3 & = & v_3 \end{array}$ or, with $r = v_3$ as a free variable, $\begin{array}{rcr} v_1 & = & -r \\ v_2 & = & -r \\ v_3 & = & r \end{array}$. So the collection of all eigenvectors of $\lambda_1 = -1$ is $\boxed{\vec{v}_1 = r\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \text{ where } r \in \mathbb{R},\ r \neq 0.}$

Page 300 Number 8 (continued 3)

Solution (continued).
$\underline{\lambda_2 = 2.}$ As above, we consider $(A - 2\mathcal{I})\vec{v}_2 = \vec{0}$ and consider the augmented matrix

$$\left[\begin{array}{ccc|c} -1-(2) & 0 & 0 & 0 \\ -4 & 2-(2) & -1 & 0 \\ 4 & 0 & 3-(2) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -4 & 0 & -1 & 0 \\ 4 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{R_1 \to R_1/(-3)}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -4 & 0 & -1 & 0 \\ 4 & 0 & 1 & 0 \end{array}\right] \underset{R_3 \to R_3 - 4R_1}{\overset{R_2 \to R_2 + 4R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3 + R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_2 \to -R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Page 300 Number 8 (continued 4)

Solution (continued). So we need $\begin{array}{rcl} v_1 & = & 0 \\ v_3 & = & 0 \\ 0 & = & 0 \end{array}$, or $\begin{array}{rcl} v_1 & = & 0 \\ v_2 & = & v_2 \\ v_3 & = & 0 \end{array}$ or, with $s = v_2$ as a free variable, $\begin{array}{rcl} v_1 & = & 0 \\ v_2 & = & s \\ v_3 & = & 0 \end{array}$. So the collection of all eigenvectors of $\lambda_2 = 2$ is $\boxed{\vec{v}_2 = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ where } s \in \mathbb{R},\ s \neq 0.}$

Page 300 Number 8 (continued 5)

Solution (continued).
$\underline{\lambda_3 = 3.}$ As above, we consider $(A - 3\mathcal{I})\vec{v}_3 = \vec{0}$ and consider the augmented matrix

$$\left[\begin{array}{ccc|c} -1-(3) & 0 & 0 & 0 \\ -4 & 2-(3) & -1 & 0 \\ 4 & 0 & 3-(3) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ -4 & -1 & -1 & 0 \\ 4 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3 + R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_2 \to -R_2}{\overset{R_1 \to R_1/(-4)}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{rcrcl} v_1 & & & = & 0 \\ v_2 & + & v_3 & = & 0 \\ & & 0 & = & 0 \end{array}$, or $\begin{array}{rcl} v_1 & = & 0 \\ v_2 & = & -v_3 \\ v_3 & = & v_3 \end{array}$ ...

Page 300 Number 8 (continued 5)

Page 300 Number 8. Find the characteristic polynomial, the real eigenvalues, and the corresponding eigenvectors for $A = \begin{bmatrix} -1 & 0 & 0 \\ -4 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix}$.

Solution (continued). ... $\begin{array}{rcl} v_1 & = & 0 \\ v_2 & = & -v_3 \\ v_3 & = & v_3 \end{array}$ or, with $t = v_3$ as a free variable, $\begin{array}{rcl} v_1 & = & 0 \\ v_2 & = & -t \\ v_3 & = & t \end{array}$. So the collection of all eigenvectors of $\lambda_3 = 3$ is

$\boxed{\vec{v}_3 = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R},\ t \neq 0.}$ □

Page 300 Number 14

Page 300 Number 14. Find the characteristic polynomial, the real eigenvalues, and the corresponding eigenvectors for $A = \begin{bmatrix} 4 & 0 & 0 \\ 8 & 4 & 8 \\ 0 & 0 & 4 \end{bmatrix}$.

Solution. We have

$$A - \lambda \mathcal{I} = \begin{bmatrix} 4 & 0 & 0 \\ 8 & 4 & 8 \\ 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-\lambda & 0 & 0 \\ 8 & 4-\lambda & 8 \\ 0 & 0 & 4-\lambda \end{bmatrix}.$$

So the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda \mathcal{I}) = \begin{vmatrix} 4-\lambda & 0 & 0 \\ 8 & 4-\lambda & 8 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (4-\lambda) \begin{vmatrix} 4-\lambda & 8 \\ 0 & 4-\lambda \end{vmatrix} - 0 + 0 = (4-\lambda)\left((4-\lambda)(4-\lambda) - (8)(0)\right)$$

$$= \boxed{(4-\lambda)^3.}$$

Page 300 Number 14 (continued 1)

Solution (continued). The eigenvalues can be found from the characteristic equation $p(\lambda) = 0$: $(4-\lambda)^3 = 0$. By Note 5.1.A, $\boxed{\text{the only eigenvalue is } \lambda = 4.}$ To find the eigenvector corresponding to $\lambda = 4$ we consider the formula $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$. This leads to the augmented matrix

$$\left[\begin{array}{ccc|c} 4-(4) & 0 & 0 & 0 \\ 8 & 4-(4) & 8 & 0 \\ 0 & 0 & 4-(4) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 8 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 8 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_1 \to R_1/8}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Page 300 Number 14 (continued 1)

Solution (continued). So we need $\begin{array}{ccccc} v_1 & + & v_3 & = & 0 \\ & & 0 & = & 0 \\ & & 0 & = & 0 \end{array}$, or

$\begin{array}{ccc} v_1 & = & -v_3 \\ v_2 & = & v_2 \\ v_3 & = & v_3 \end{array}$. With $r = v_2$ and $s = v_3$ as free variables, $\begin{array}{ccc} v_1 & = & -s \\ v_2 & = & r \\ v_3 & = & s \end{array}$.

So the collection of all eigenvectors of $\lambda = 4$ is

$$\boxed{\vec{v} = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ where } r, s \in \mathbb{R} \text{ and not both } r = 0 \text{ and } s = 0.}$$

□

Theorem 5.1

Theorem 5.1. Properties of Eigenvalues and Eigenvectors.
Let A be an $n \times n$ matrix.

2. If λ is an eigenvalue of an invertible matrix A with $\vec{v}$ as a corresponding eigenvector, then $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of A^{-1}, again with $\vec{v}$ as a corresponding eigenvector.

Proof. Page 301 Number 28. If $\lambda = 0$ is an eigenvalue of matrix A then there is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v} = \vec{0}$. But then the homogeneous system of equations associated with $A\vec{v} = \vec{0}$ has a nontrivial solution. This implies that A is not invertible (by Theorem 1.16). But λ is given to be an eigenvalue of an invertible matrix, so it must be that, in fact, $\lambda \neq 0$. If λ is an eigenvalue of A with eigenvector $\vec{v}$, then $A\vec{v} = \lambda\vec{v}$. Therefore $A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$ or, by Theorem 1.3.A(7), "Scalars Pull Through," $\vec{v} = \lambda A^{-1}\vec{v}$. So $A^{-1}\vec{v} = (1/\lambda)\vec{v}$ and $1/\lambda$ is an eigenvalue of A^{-1}. □

Page 298 Example 8

Page 298 Example 8. Let D_∞ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ and having derivatives of all order. Let $T : D_\infty \to D_\infty$ be the differentiation map so that $T(f) = f'$. Describe all eigenvalues and eigenvectors of T. (Notice that by Example 3.4.5, T actually is linear.)
Solution. We need scalars λ and nonzero functions f where $T(f) = \lambda f$.
Case 1. If $\lambda = 0$, then we need $T(f) = 0f = 0$ or $f' = 0$. So f must be a constant function. Eigenvectors are nonzero by definition, so the eigenvectors associated with eigenvalue 0 are
$\boxed{\text{all } f(x) = k \text{ where } k \in \mathbb{R},\ k \neq 0.}$
Case 2. If $\lambda \neq 0$, then we need $T(f) = \lambda f$ or $f' = \lambda f$. That is, $dy/dx = \lambda y$ or (as a separable differential equation), $dy/y = \lambda\, dx$ and so $\int \frac{1}{y}\, dy = \int \lambda\, dx$ or $\ln|y| = \lambda x + c$ or $e^{\ln|y|} = e^{\lambda x + c}$ or $|y| = e^c e^{\lambda x}$ or $y = \pm e^c e^{\lambda x}$ or $y = ke^{\lambda x}$ where we set $k = e^c$ or $k = -e^c$ (so $k \neq 0$). So the eigenvectors associated with eigenvalue $\lambda \neq 0$ are
$\boxed{\text{all } y = ke^{\lambda x} \text{ where } k \neq 0.}$ □

Page 300 Number 18

Page 300 Number 18. Find the eigenvalues and corresponding eigenvectors for the linear transformation $T([x, y]) = [x - y, -x + y])$.
Solution. We apply Corollary 2.3.A, "Standard Matrix Representation of Linear Transformations," to find the matrix representing T. We have $T(\hat{\imath}) = T([1, 0]) = [(1) - (0), (-1) + (0)] = [1, -1]$ and $T(\hat{\jmath}) = T([0, 1]) = [(0) - (1), -(0) + (1)] = [-1, 1]$. Hence the standard matrix representation of T is $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. By Note 5.1.B, the eigenvalues and eigenvectors of T are the same as those of A. So we consider the characteristic polynomial

$$p(\lambda) = \det(A - \lambda\mathcal{I}) = \det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda) - (-1)(-1) = 1 - 2\lambda + \lambda^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2).$$

We find the eigenvalues from the characteristic polynomial $p(\lambda) = \lambda(\lambda - 2) = 0$. So $\boxed{\text{the eigenvalues of } T \text{ are } \lambda_1 = 0 \text{ and } \lambda_2 = 2.}$

Page 300 Number 18 (continued 1)

Solution (continued). Denote the eigenvalues as $\lambda_1 = 0$ and $\lambda_2 = 2$. To find the eigenvectors corresponding to each eigenvalue, we consider the formula $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$.
$\lambda_1 = 0$. With $\vec{v}_1 = [v_1, v_2]^T$ an eigenvector corresponding to eigenvalue $\lambda_1 = 0$ we need $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$. So we consider the augmented matrix

$$\left[\begin{array}{cc|c} 1-(0) & -1 & 0 \\ -1 & 1-(0) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \overset{R_2 \to R_2 + R_1}{\widetilde{\quad}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{rcl} v_1 - v_2 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcl} v_1 & = & v_2 \\ v_2 & = & v_2 \end{array}$ or, with $r = v_2$ as a free variable, $\begin{array}{rcl} v_1 & = & r \\ v_2 & = & r \end{array}$. So the collection of all eigenvalues of $\lambda_1 = 0$ is

$$\boxed{\vec{v}_1 = r \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } r \in \mathbb{R},\ r \neq 0.}$$

Page 300 Number 18 (continued 2)

Solution (continued).
$\lambda_2 = 2$. As above, we need $(A - 2\mathcal{I})\vec{v}_2 = \vec{0}$ and consider the augmented matrix

$$\left[\begin{array}{cc|c} 1-(2) & -1 & 0 \\ -1 & 1-(2) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$\overset{R_2 \to R_2 - R_1}{\widetilde{\quad}} \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \overset{R_1 \to -R_1}{\widetilde{\quad}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{rcl} v_1 + v_2 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcl} v_1 & = & -v_2 \\ v_2 & = & v_2 \end{array}$ or with $s = v_2$ as a free variable, $\begin{array}{rcl} v_1 & = & -s \\ v_2 & = & s \end{array}$. So the collection of all eigenvalues of $\lambda_2 = 2$ is

$$\boxed{\vec{v}_2 = s \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ where } s \in \mathbb{R},\ s \neq 0.}$$

Page 301 Number 30

Page 301 Number 30. Prove that a square matrix is invertible if and only if no eigenvalue is zero.

Proof. Suppose A is invertible. Then by Theorem 4.3, "Determinant Criterion for Invertibility," $\det(A) \neq 0$. Now if $\lambda = 0$ is an eigenvalue then

$$\det(A - \lambda \mathcal{I}) = \det(A - 0\mathcal{I}) = \det(A) = 0,$$

so 0 cannot be an eigenvalue.

Suppose $\lambda = 0$ is an eigenvalue. Then, again,

$$\det(A - \lambda \mathcal{I}) = \det(A - 0\mathcal{I}) = \det(A) = 0.$$

So by Theorem 4.3, A is not invertible. □

Page 301 Number 32

Page 301 Number 32. Let A be an $n \times n$ matrix and let $\mathcal{I}$ be the $n \times n$ identity matrix. Compare the eigenvectors and eigenvalues of A with those of $A + r\mathcal{I}$ for a scalar r.

Solution. Suppose λ is an eigenvalue of A with corresponding eigenvector $\vec{v}$. Then $A\vec{v} = \lambda\vec{v}$. So

$$(A + r\mathcal{I})\vec{v} = A\vec{v} + r\mathcal{I}\vec{v} = A\vec{v} + r\vec{v} = \lambda\vec{v} + r\vec{v} = (\lambda + r)\vec{v}.$$

So $\lambda + r$ is an eigenvalue of $A + r\mathcal{I}$ with $\vec{v}$ as a corresponding eigenvector. Conversely, if $\lambda + r$ is an eigenvalue of $A + r\mathcal{I}$ with eigenvector $\vec{w}$ then $(A + r\mathcal{I})\vec{w} = (\lambda + r)\vec{w}$ or $A\vec{w} + r\vec{w} = \lambda\vec{w} + r\vec{w}$ or $A\vec{w} = \lambda\vec{w}$ so $\vec{w}$ is an eigenvector of A corresponding to eigenvalues λ.

So the eigenvalues of $A + r\mathcal{I}$ are precisely those of the form $\lambda + r$ where λ is an eigenvalue of A. The corresponding eigenvectors of $A + r\mathcal{I}$ corresponding to $\lambda + r$ are precisely the eigenvectors of A corresponding to λ. □

Page 302 Number 38

Page 302 Number 38. Let A be an $n \times n$ matrix and let C be an invertible $n \times n$ matrix. Prove that the eigenvalues of A and of $C^{-1}AC$ are the same.

Solution. Notice that

$$\begin{aligned} C^{-1}AC - \lambda\mathcal{I} &= C^{-1}AC - \lambda C^{-1}C \\ &= C^{-1}AC - C^{-1}(\lambda C) \text{ by Theorem 1.3.A(7),} \\ &\quad \text{“Scalars Pull Through”} \\ &= C^{-1}(AC - \lambda C) \text{ by Theorem 1.3.A(10),} \\ &\quad \text{“Distribution Law of Matrix Multiplication”} \\ &= C^{-1}(A - \lambda\mathcal{I})C \text{ by Theorem 1.3.A(10).} \end{aligned}$$

Page 302 Number 38 (continued)

Solution (continued). Recall that $\det(C^{-1}) = 1/\det(C)$ by Exercise 4.2.31. So the characteristic polynomial for $C^{-1}AC$ is

$$\begin{aligned} \det(C^{-1}AC - \lambda\mathcal{I}) &= \det(C^{-1}(A - \lambda\mathcal{I})C) \text{ as just shown} \\ &= \det(C^{-1})\det(A - \lambda\mathcal{I})\det(C) \text{ by Theorem 4.4,} \\ &\quad \text{“The Multiplicative Property”} \\ &= (1/\det(C))\det(A - \lambda\mathcal{I})\det(C) \\ &= \det(A - \lambda\mathcal{I}). \end{aligned}$$

Now $\det(A - \lambda\mathcal{I})$ is the characteristic polynomial of A, so A and $C^{-1}AC$ have the same characteristic polynomials. These polynomials have the same roots (of course) and since the eigenvalues of a matrix are the roots of the characteristic polynomial (see Note 5.1.A), A and $C^{-1}AC$ have the same eigenvalues, as claimed. □

Page 302 Number 40

Page 302 Number 40. The Cayley-Hamilton Theorem states:

Cayley-Hamilton Theorem. Every square matrix A satisfies its characteristic equation. That is, if $p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ is the characteristic polynomial of A then $p(A) = a_nA^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0\mathcal{I} = O$ (where O is the $n \times n$ zero matrix).

Use the Cayley-Hamilton Theorem to prove that, for invertible $n \times n$ matrix A, A^{-1} can be computed as a linear combination of $A^0 = \mathcal{I}, A, A^2, \ldots, A^{n-1}$.

Proof. Let A be an invertible $n \times n$ matrix and let $p(\lambda)$ be the characteristic polynomial of A. Then by the Cayley-Hamilton Theorem,

$$p(A) = a_nA^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0\mathcal{I} = O.$$

So $a_nA^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A = -a_0\mathcal{I}$. Multiplying both sides of this equation on the right be A^{-1} gives

$$(a_nA^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A)A^{-1} = (-a_0\mathcal{I})A^{-1} \ldots$$

Page 302 Number 40 (continued)

Proof (continued). . . . or, by Theorem 1.3.A(1), "Distribution Law of Matrix Multiplication,"

$$a_nA^nA^{-1} + a_{n-1}A^{n-1}A^{-1} + \cdots + a_2A^2A^{-1} + a_1AA^{-1} = (-a_0\mathcal{I})A^{-1}$$

or by Theorem 1.3.A(10), "Associativity Law of Matrix Multiplication," and Theorem 1.3.A(6), "Associative Law of Matrix Multiplication,"

$$a_nA^{n-1}(AA^{-1}) + a_{n-1}A^{n-2}(AA^{-1}) + \cdots + a_2A(AA^{-1}) + a_1(AA^{-1}) = -a_0\mathcal{I}A^{-1}$$

or

$$a_nA^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2A + a_1\mathcal{I} = -a_0A^{-1}.$$

Since A is invertible, then 0 is not an eigenvalue of A by Exercise 30, so $p(0) = a_0 \neq 0$. We then have

$$A^{-1} = -\frac{a_n}{a_0}A^{n-1} - \frac{a_{n-1}}{a_0}A^{n-2} - \cdots - \frac{a_2}{a_0}A - \frac{a_1}{a_0}\mathcal{I}.$$

So A^{-1} is a linear combination of $A^{n-1}, A^{n-2}, \ldots, A, \mathcal{I}$, as claimed. □