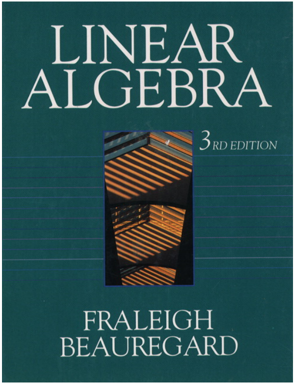
Linear Algebra

Chapter 5: Eigenvalues and Eigenvectors

Section 5.2. Diagonalization—Proofs of Theorems



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Theorem 5.2

Theorem 5.2. Matrix Summary of Eigenvalues of A.
Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be (possibly complex) scalars and $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ be nonzero vectors in n-space. Let C be the $n \times n$ matrix having $\vec{v_j}$ as jth column vector and let

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Then $AC = CD$ if and only if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A and $\vec{v_j}$ is an eigenvector of A corresponding to λ_j for $j = 1, 2, \ldots, n$.

Theorem 5.2 (continued)

Proof. We have

$$CD = \begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} \vdots & \vdots & & \vdots \\ \lambda_1\vec{v_1} & \lambda_2\vec{v_2} & \cdots & \lambda_n\vec{v_n} \\ \vdots & \vdots & & \vdots \end{bmatrix}.$$

Also, $AC = A \begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \\ \vdots & \vdots & & \vdots \end{bmatrix}$. Therefore, $AC = CD$ if and only if $A\vec{v_j} = \lambda_j \vec{v_j}$. □

Corollary 1

Corollary 1. A Criterion for Diagonalization.
An $n \times n$ matrix A is diagonalizable if and only if n-space has a basis consisting of eigenvectors of A.

Proof. Suppose A is diagonalizable. Then by Definition 5.3, "Diagonalizable Matrix," $C^{-1}AC = D$ for some invertible $n \times n$ matrix C.

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Corollary 1 (continued)

Proof (continued). Since n-space is dimension n and the column vectors of C form a set of n vectors which span n-space then the vectors must be linearly independent (by Theorem 2.3(3a), "Existence and Determination of Bases") and so are a basis for n-space by Definition 3.6, "Basis of a Vector Space."

Conversely, suppose n-space has a basis consisting of eigenvectors of A, say $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ where $\vec{v}_j$ is an eigenvector corresponding to eigenvalue λ_j. Then by Definition 3.6, "Basis of a Vector Space," the vectors are linearly independent. So if we form matrix C where the jth column of C is $\vec{v}_j$ then C is invertible by Theorem 1.12, "Conditions for A^{-1} to Exist."

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Corollary 2

Corollary 2. Computation of A^k.
Let an $n \times n$ matrix A have n eigenvectors and eigenvalues, giving rise to the matrices C and D so that $AC = CD$, as described in Theorem 5.2. If the eigenvectors are independent, then C is an invertible matrix and $C^{-1}AC = D$. Under these conditions, we have $A^k = CD^kC^{-1}$.

Proof. By Corollary 1, if the eigenvectors of A are independent, then A is diagonalizable and so C is invertible. Now consider

$$\begin{aligned} A^k &= \underbrace{(CDC^{-1})(CDC^{-1})\cdots(CDC^{-1})}_{k \text{ factors}} \\ &= CD(C^{-1}C)D(C^{-1}C)D(C^{-1}C)\cdots(C^{-1}C)DC^{-1} \\ &= CD\mathcal{I}D\mathcal{I}D\cdots\mathcal{I}DC^{-1} \\ &= C\underbrace{DDD\cdots D}_{k \text{ factors}}C^{-1} = CD^kC^{-1} \end{aligned}$$

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□

Example 5.2.A

Example 5.2.A. Diagonalize $A = \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix}$ and calculate A^k.

Solution. We have
$A - \lambda\mathcal{I} = \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5-\lambda & -3 \\ -6 & 2-\lambda \end{bmatrix}$. So the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda\mathcal{I}) = \begin{vmatrix} 5-\lambda & -3 \\ -6 & 2-\lambda \end{vmatrix}$$

$$= (5-\lambda)(2-\lambda)-(-3)(-6) = 10-7\lambda+\lambda^2-18 = \lambda^2-7\lambda-8 = (\lambda+1)(\lambda-8).$$

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So the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 8$. To find the eigenvectors corresponding to each eigenvalue we consider the formula $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$.

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Example 5.2.A (continued 1)

Solution (continued).
$\underline{\lambda_1 = -1.}$ With $\vec{v}_1 = [v_1, v_2]$ an eigenvector corresponding to the eigenvalue $\lambda_1 = -1$ we need $(A - (-1)\mathcal{I})\vec{v}_1 = \vec{0}$. So we consider the augmented matrix

$$\left[\begin{array}{cc|c} 5-(-1) & -3 & 0 \\ -6 & 2-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 6 & -3 & 0 \\ -6 & 3 & 0 \end{array}\right] \overset{R_2 \to R_2 + R_1}{\widetilde{}} \left[\begin{array}{cc|c} 6 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

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So we need $\begin{array}{rcl} 6v_1 - 3v_2 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcl} v_1 & = & (1/2)v_2 \\ v_2 & = & v_2 \end{array}$ or, with $r = v_2/2$ as a free variable, $\begin{array}{rcl} v_1 & = & r \\ v_2 & = & 2r \end{array}$. So the collection of all eigenvectors of $\lambda_1 = -1$ is $\vec{v}_1 = r\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where $r \in \mathbb{R}$, $r \neq 0$.

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Solution (continued).

$\underline{\lambda_2 = 8.}$ As above, we consider $(A - 8\mathcal{I})\vec{v}_2 = \vec{0}$ and consider the augmented matrix

$$\left[\begin{array}{cc|c} 5-(8) & -3 & 0 \\ -6 & 2-(8) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -3 & -3 & 0 \\ -6 & -6 & 0 \end{array}\right]$$

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Example 5.2.A (continued 3)

Solution (continued). If we take $r = s = 1$ then we have the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 8$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively. So by Theorem 5.2, "Matrix Summary of Eigenvalues of A," with $C = \begin{bmatrix} \vdots & \vdots \\ \vec{v}_1 & \vec{v}_2 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}$ we have $AC = CD$. We find C^{-1} (by Note 1.5.A, "Computation of Inverses"):

$$[C|\mathcal{I}] = \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \overset{R_2 \to R_2 - 2R_1}{\widetilde{}} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{array}\right]$$

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Example 5.2.A (continued 4)

Solution (continued).

$$\left[\begin{array}{rr|rr} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{array}\right] \underbrace{}_{R_2 \to R_2/3} \left[\begin{array}{rr|rr} 1 & -1 & 1 & 0 \\ 0 & 1 & -2/3 & 1/3 \end{array}\right]$$

$$\underbrace{}_{R_1 \to R_1 + R_2} \left[\begin{array}{rr|rr} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & -2/3 & 1/3 \end{array}\right],$$

so $C^{-1} = \left[\begin{array}{rr} 1/3 & 1/3 \\ -2/3 & 1/3 \end{array}\right] = \frac{1}{3}\left[\begin{array}{rr} 1 & 1 \\ -2 & 1 \end{array}\right]$. So $A = CDC^{-1}$ where

$$C = \left[\begin{array}{rr} 1 & -1 \\ 2 & 1 \end{array}\right],\ D = \left[\begin{array}{rr} -1 & 0 \\ 0 & 8 \end{array}\right],\ \text{and } C^{-1} = \frac{1}{3}\left[\begin{array}{rr} 1 & 1 \\ -2 & 1 \end{array}\right].$$

Example 5.2.A (continued 4)

Solution (continued).

$$\left[\begin{array}{rr|rr} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{array}\right] \underbrace{}_{R_2 \to R_2/3} \left[\begin{array}{rr|rr} 1 & -1 & 1 & 0 \\ 0 & 1 & -2/3 & 1/3 \end{array}\right]$$

$$\underbrace{}_{R_1 \to R_1 + R_2} \left[\begin{array}{rr|rr} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & -2/3 & 1/3 \end{array}\right],$$

so $C^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$. So $\boxed{A = CDC^{-1} \text{ where}}$

$$\boxed{C = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}, \text{ and } C^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}.}$$

Example 5.2.A (continued 5)

Solution (continued). As in Corollary 2,

$$\begin{aligned}
A^k &= CD^kC^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}^k \left(\frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \right) \\
&= \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 \\ 0 & 8^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} (-1)^k & -8^k \\ 2(-1)^k & 8^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \\
&= \boxed{\tfrac{1}{3} \begin{bmatrix} (-1)^k + 2(8^k) & (-1)^k - 8^k \\ 2(-1)^k - 2(8^k) & 2(-1)^k + 8^k \end{bmatrix}}.
\end{aligned}$$

□

Theorem 5.3

Theorem 5.3. Independence of Eigenvectors.
Let A be an $n \times n$ matrix. If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is linearly independent and A is diagonalizable.

Proof. We prove this by contradiction. Suppose that the conclusion is false and the hypotheses are true. That is, suppose the eigenvectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are linearly dependent.

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Proof. We prove this by contradiction. Suppose that the conclusion is false and the hypotheses are true. That is, suppose the eigenvectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are linearly dependent. Then one of them is a linear combination of its predecessors (see page 203 number 37). Let $\vec{v_k}$ be the first such vector, so that

$$\vec{v_k} = d_1\vec{v_1} + d_2\vec{v_2} + \cdots + d_{k-1}\vec{v}_{k-1} \qquad (2)$$

and $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v}_{k-1}\}$ is independent.

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Theorem 5.3. Independence of Eigenvectors.
Let A be an $n \times n$ matrix. If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is linearly independent and A is diagonalizable.

Proof. We prove this by contradiction. Suppose that the conclusion is false and the hypotheses are true. That is, suppose the eigenvectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are linearly dependent. Then one of them is a linear combination of its predecessors (see page 203 number 37). Let $\vec{v_k}$ be the first such vector, so that

$$\vec{v_k} = d_1\vec{v_1} + d_2\vec{v_2} + \cdots + d_{k-1}\vec{v}_{k-1} \qquad (2)$$

and $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v}_{k-1}\}$ is independent. Multiplying (2) by λ_k, we obtain

$$\lambda_k\vec{v_k} = d_1\lambda_k\vec{v_1} + d_2\lambda_k\vec{v_2} + \cdots + d_{k-1}\lambda_k\vec{v}_{k-1}. \qquad (3)$$

Theorem 5.3

Theorem 5.3. Independence of Eigenvectors.
Let A be an $n \times n$ matrix. If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is linearly independent and A is diagonalizable.

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Theorem 5.3 (continued)

Theorem 5.3. Independence of Eigenvectors.
Let A be an $n \times n$ matrix. If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is linearly independent and A is diagonalizable.

Proof (continued). Also, multiplying (2) on the left by the matrix A yields

$$\lambda_k \vec{v_k} = d_1 \lambda_1 \vec{v_1} + d_2 \lambda_2 \vec{v_2} + \cdots + d_{k-1} \lambda_{k-1} \vec{v}_{k-1} \qquad (4),$$

since $A\vec{v_i} = \lambda_i \vec{v_i}$. Subtracting (4) from (3), we see that

$$\vec{0} = d_1(\lambda_k - \lambda_1)\vec{v_1} + d_2(\lambda_k - \lambda_2)\vec{v_2} + \cdots + d_{k-1}(\lambda_k - \lambda_{k-1})\vec{v}_{k-1}.$$

But this equation is a dependence relation since not all d_i's are 0 and the λ's are hypothesized to be different. This contradicts the linear independence of the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v}_{k-1}\}$. This contradiction shows that $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is independent. From Corollary 1 of Theorem 5.2 we see that A is diagonalizable. □

Theorem 5.3 (continued)

Theorem 5.3. Independence of Eigenvectors.
Let A be an $n \times n$ matrix. If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is linearly independent and A is diagonalizable.

Proof (continued). Also, multiplying (2) on the left by the matrix A yields

$$\lambda_k \vec{v_k} = d_1\lambda_1\vec{v_1} + d_2\lambda_2\vec{v_2} + \cdots + d_{k-1}\lambda_{k-1}\vec{v}_{k-1} \qquad (4),$$

since $A\vec{v_i} = \lambda_i\vec{v_i}$. Subtracting (4) from (3), we see that

$$\vec{0} = d_1(\lambda_k - \lambda_1)\vec{v_1} + d_2(\lambda_k - \lambda_2)\vec{v_2} + \cdots + d_{k-1}(\lambda_k - \lambda_{k-1})\vec{v}_{k-1}.$$

But this equation is a dependence relation since not all d_i's are 0 and the λ's are hypothesized to be different. This contradicts the linear independence of the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v}_{k-1}\}$. This contradiction shows that $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is independent. From Corollary 1 of Theorem 5.2 we see that A is diagonalizable. □

Page 315 Number 6

Page 315 Number 6. Find the eigenvalues λ_i and corresponding eigenvectors $\vec{v}_i$ of $A = \begin{bmatrix} -3 & 5 & -20 \\ 2 & 0 & 8 \\ 2 & 1 & 7 \end{bmatrix}$. Find an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$.

Solution. We show all computations and details, so this will take a while. . .

Page 315 Number 6

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Solution. We show all computations and details, so this will take a while. . .

We have

$$A - \lambda\mathcal{I} = \begin{bmatrix} -3 & 5 & -20 \\ 2 & 0 & 8 \\ 2 & 1 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3-\lambda & 5 & -20 \\ 2 & 0-\lambda & 8 \\ 2 & 1 & 7-\lambda \end{bmatrix}.$$

Page 315 Number 6

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We have

$$A - \lambda\mathcal{I} = \begin{bmatrix} -3 & 5 & -20 \\ 2 & 0 & 8 \\ 2 & 1 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3-\lambda & 5 & -20 \\ 2 & 0-\lambda & 8 \\ 2 & 1 & 7-\lambda \end{bmatrix}.$$

Page 315 Number 6 (continued 1)

Solution (continued). So the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda\mathcal{I}) = \begin{vmatrix} -3-\lambda & 5 & -20 \\ 2 & 0-\lambda & 8 \\ 2 & 1 & 7-\lambda \end{vmatrix}$$

$$= (-3-\lambda)\begin{vmatrix} -\lambda & 8 \\ 1 & 7-\lambda \end{vmatrix} - (5)\begin{vmatrix} 2 & 8 \\ 2 & 7-\lambda \end{vmatrix} + (-20)\begin{vmatrix} 2 & -\lambda \\ 2 & 1 \end{vmatrix}$$

$$= (-3-\lambda)\left((-\lambda)(7-\lambda) - (8)(1)\right) - 5\left((2)(7-\lambda) - (8)(2)\right)$$

$$-20\left((2)(1) - (-\lambda)(2)\right) = (-3-\lambda)(\lambda^2 - 7\lambda - 8) - 5(-2\lambda - 2) - 20(2\lambda + 2)$$

Page 315 Number 6 (continued 1)

Solution (continued). So the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda\mathcal{I}) = \begin{vmatrix} -3-\lambda & 5 & -20 \\ 2 & 0-\lambda & 8 \\ 2 & 1 & 7-\lambda \end{vmatrix}$$

$$= (-3-\lambda)\begin{vmatrix} -\lambda & 8 \\ 1 & 7-\lambda \end{vmatrix} - (5)\begin{vmatrix} 2 & 8 \\ 2 & 7-\lambda \end{vmatrix} + (-20)\begin{vmatrix} 2 & -\lambda \\ 2 & 1 \end{vmatrix}$$

$$= (-3-\lambda)\left((-\lambda)(7-\lambda) - (8)(1)\right) - 5\left((2)(7-\lambda) - (8)(2)\right)$$

$$-20\left((2)(1) - (-\lambda)(2)\right) = (-3-\lambda)(\lambda^2 - 7\lambda - 8) - 5(-2\lambda - 2) - 20(2\lambda + 2)$$

$$= (-3-\lambda)(\lambda - 8)(\lambda + 1) + 10(\lambda + 1) - 40(\lambda + 1)$$

$$= (\lambda+1)\left((-3-\lambda)(\lambda-8) + 10 - 40\right) = (\lambda+1)(-3\lambda+24-\lambda^2+8\lambda+10-40)$$

$$= (\lambda+1)(-\lambda^2+5\lambda-6) = -(\lambda+1)(\lambda^2-5\lambda+6) = -(\lambda+1)(\lambda-2)(\lambda-3).$$

Page 315 Number 6 (continued 1)

Solution (continued). So the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda\mathcal{I}) = \begin{vmatrix} -3-\lambda & 5 & -20 \\ 2 & 0-\lambda & 8 \\ 2 & 1 & 7-\lambda \end{vmatrix}$$

$$= (-3-\lambda)\begin{vmatrix} -\lambda & 8 \\ 1 & 7-\lambda \end{vmatrix} - (5)\begin{vmatrix} 2 & 8 \\ 2 & 7-\lambda \end{vmatrix} + (-20)\begin{vmatrix} 2 & -\lambda \\ 2 & 1 \end{vmatrix}$$

$$= (-3-\lambda)\left((-\lambda)(7-\lambda) - (8)(1)\right) - 5\left((2)(7-\lambda) - (8)(2)\right)$$
$$-20\left((2)(1) - (-\lambda)(2)\right) = (-3-\lambda)(\lambda^2 - 7\lambda - 8) - 5(-2\lambda - 2) - 20(2\lambda + 2)$$

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$$= (\lambda+1)\left((-3-\lambda)(\lambda-8) + 10 - 40\right) = (\lambda+1)(-3\lambda+24-\lambda^2+8\lambda+10-40)$$
$$= (\lambda+1)(-\lambda^2+5\lambda-6) = -(\lambda+1)(\lambda^2-5\lambda+6) = -(\lambda+1)(\lambda-2)(\lambda-3).$$

Page 315 Number 6 (continued 2)

Solution (continued). Since $p(\lambda) = -(\lambda+1)(\lambda-2)(\lambda-3)$, then the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 3$. To find the eigenvectors corresponding to each eigenvalue we consider the formula $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$.

$\underline{\lambda_1 = -1.}$ With $\vec{v}_1 = [v_1, v_2, v_3]$ an eigenvector corresponding to eigenvalue $\lambda_1 = -1$ we need $(A - \lambda\mathcal{I})\vec{v}_1 = \vec{0}$.

Page 315 Number 6 (continued 2)

Solution (continued). Since $p(\lambda) = -(\lambda+1)(\lambda-2)(\lambda-3)$, then the $\boxed{\text{eigenvalues are } \lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 3.}$ To find the eigenvectors corresponding to each eigenvalue we consider the formula $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$.

$\underline{\lambda_1 = -1.}$ With $\vec{v}_1 = [v_1, v_2, v_3]$ an eigenvector corresponding to eigenvalue $\lambda_1 = -1$ we need $(A - \lambda\mathcal{I})\vec{v}_1 = \vec{0}$. So we consider the augmented matrix

$$\left[\begin{array}{ccc|c} -3-(-1) & 5 & -20 & 0 \\ 2 & 0-(-1) & 8 & 0 \\ 2 & 1 & 7-(-1) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -2 & 5 & -20 & 0 \\ 2 & 1 & 8 & 0 \\ 2 & 1 & 8 & 0 \end{array}\right]$$

$$\underbrace{R_3 \to R_3 - R_2}\ \left[\begin{array}{ccc|c} -2 & 5 & -20 & 0 \\ 2 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underbrace{R_2 \to R_2 + R_1}\ \left[\begin{array}{ccc|c} -2 & 5 & -20 & 0 \\ 0 & 6 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Page 315 Number 6 (continued 2)

Solution (continued). Since $p(\lambda) = -(\lambda+1)(\lambda-2)(\lambda-3)$, then the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 3$. To find the eigenvectors corresponding to each eigenvalue we consider the formula $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$.

$\underline{\lambda_1 = -1.}$ With $\vec{v}_1 = [v_1, v_2, v_3]$ an eigenvector corresponding to eigenvalue $\lambda_1 = -1$ we need $(A - \lambda\mathcal{I})\vec{v}_1 = \vec{0}$. So we consider the augmented matrix

$$\left[\begin{array}{ccc|c} -3-(-1) & 5 & -20 & 0 \\ 2 & 0-(-1) & 8 & 0 \\ 2 & 1 & 7-(-1) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -2 & 5 & -20 & 0 \\ 2 & 1 & 8 & 0 \\ 2 & 1 & 8 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3 - R_2}{\sim} \left[\begin{array}{ccc|c} -2 & 5 & -20 & 0 \\ 2 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_2 \to R_2 + R_1}{\sim} \left[\begin{array}{ccc|c} -2 & 5 & -20 & 0 \\ 0 & 6 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Page 315 Number 6 (continued 3)

Solution (continued).

$$\left[\begin{array}{rrr|r} -2 & 5 & -20 & 0 \\ 0 & 6 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underbrace{R_2 \to R_2/6}_{\sim} \left[\begin{array}{rrr|r} -2 & 5 & -20 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underbrace{R_1 \to R_1 - 5R_2}_{\sim} \left[\begin{array}{rrr|r} -2 & 0 & -10 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underbrace{R_1 \to R_1/(-2)}_{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{rcrcl} v_1 & & + & 5v_3 & = & 0 \\ & v_2 & - & 2v_3 & = & 0 \\ & & & 0 & = & 0 \end{array}$ or $\begin{array}{rcr} v_1 & = & -5v_3 \\ v_2 & = & 2v_3 \\ v_3 & = & v_3 \end{array}$ or, with $r = v_3$ as a free variable, $\begin{array}{rcr} v_1 & = & -5r \\ v_2 & = & 2r \\ v_3 & = & r \end{array}$.

Page 315 Number 6 (continued 3)

Solution (continued).

$$\left[\begin{array}{rrr|r} -2 & 5 & -20 & 0 \\ 0 & 6 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underbrace{}_{R_2 \to R_2/6} \left[\begin{array}{rrr|r} -2 & 5 & -20 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underbrace{}_{R_1 \to R_1 - 5R_2} \left[\begin{array}{rrr|r} -2 & 0 & -10 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underbrace{}_{R_1 \to R_1/(-2)} \left[\begin{array}{rrr|r} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{rcrcl} v_1 & & +5v_3 & = & 0 \\ & v_2 & -2v_3 & = & 0 \\ & & 0 & = & 0 \end{array}$ or $\begin{array}{rcr} v_1 & = & -5v_3 \\ v_2 & = & 2v_3 \\ v_3 & = & v_3 \end{array}$ or, with $r = v_3$ as a free variable, $\begin{array}{rcr} v_1 & = & -5r \\ v_2 & = & 2r \\ v_3 & = & r \end{array}$.

Page 315 Number 6 (continued 4)

Solution (continued). So the collection of all eigenvectors of $\lambda_1 = -1$ is

$$\vec{v}_1 = r \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \text{ where } r \in \mathbb{R},\ r \neq 0.$$

$\underline{\lambda_2 = 2.}$ As above, we consider $(A - 2\mathcal{I})\vec{v}_2 = \vec{0}$ and consider the augmented matrix

$$\left[\begin{array}{ccc|c} -3-(2) & 5 & -20 & 0 \\ 2 & 0-(2) & 8 & 0 \\ 2 & 1 & 7-(2) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -5 & 5 & -20 & 0 \\ 2 & -2 & 8 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right]$$

$$\underset{R_2 \to R_2/2}{\overset{R_1 \to R_1/(-5)}{\widetilde{}}} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 1 & -1 & 4 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right] \underset{R_3 \to R_3 - 2R_1}{\overset{R_2 \to R_2 - R_1}{\widetilde{}}} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

Page 315 Number 6 (continued 4)

Solution (continued). So the collection of all eigenvectors of $\lambda_1 = -1$ is

$$\boxed{\vec{v}_1 = r \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \text{ where } r \in \mathbb{R},\ r \neq 0.}$$

$\underline{\lambda_2 = 2.}$ As above, we consider $(A - 2\mathcal{I})\vec{v}_2 = \vec{0}$ and consider the augmented matrix

$$\left[\begin{array}{ccc|c} -3-(2) & 5 & -20 & 0 \\ 2 & 0-(2) & 8 & 0 \\ 2 & 1 & 7-(2) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -5 & 5 & -20 & 0 \\ 2 & -2 & 8 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right]$$

$$\underset{R_2 \to R_2/2}{\overbrace{R_1 \to R_1/(-5)}} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 1 & -1 & 4 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right] \underset{R_3 \to R_3 - 2R_1}{\overbrace{R_2 \to R_2 - R_1}} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

Page 315 Number 6 (continued 5)

Solution (continued).

$$\left[\begin{array}{rrr|r} 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \underbrace{R_3 \to R_3/3}{} \left[\begin{array}{rrr|r} 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\underbrace{R_2 \leftrightarrow R_3}{} \left[\begin{array}{rrr|r} 1 & -1 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underbrace{R_1 \to R_1 + R_2}{} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{rcrcl} v_1 & & + & 3v_3 & = & 0 \\ & v_2 & - & v_3 & = & 0 \\ & & & 0 & = & 0 \end{array}$ or $\begin{array}{rcr} v_1 & = & -3v_3 \\ v_2 & = & v_3 \\ v_3 & = & v_3 \end{array}$ or, with $s = v_3$ as a free variable, $\begin{array}{rcr} v_1 & = & -3s \\ v_2 & = & s \\ v_3 & = & s \end{array}$.

Page 315 Number 6 (continued 5)

Solution (continued).

$$\left[\begin{array}{rrr|r} 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \underbrace{R_3 \to R_3/3} \left[\begin{array}{rrr|r} 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\underbrace{R_2 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 1 & -1 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underbrace{R_1 \to R_1 + R_2} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{rcrcl} v_1 & & + 3v_3 & = & 0 \\ & v_2 & - v_3 & = & 0 \\ & & 0 & = & 0 \end{array}$ or $\begin{array}{rcr} v_1 & = & -3v_3 \\ v_2 & = & v_3 \\ v_3 & = & v_3 \end{array}$ or, with $s = v_3$ as a free variable, $\begin{array}{rcr} v_1 & = & -3s \\ v_2 & = & s \\ v_3 & = & s \end{array}$.

Page 315 Number 6 (continued 6)

Solution (continued). So the collection of all eigenvectors of $\lambda_2 = 2$ is

$$\vec{v}_2 = s \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \text{ where } s \in \mathbb{R},\ s \neq 0.$$

$\underline{\lambda_3 = 3.}$ As above, we consider $(A - 3\mathcal{I})\vec{v}_3 = \vec{0}$ and consider the augmented matrix

$$\left[\begin{array}{ccc|c} -3-(3) & 5 & -20 & 0 \\ 2 & 0-(3) & 8 & 0 \\ 2 & 1 & 7-(3) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -6 & 5 & -20 & 0 \\ 2 & -3 & 8 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right]$$

$$\underset{\smile}{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 2 & -3 & 8 & 0 \\ -6 & 5 & -20 & 0 \end{array}\right] \overset{R_2 \to R_2 - R_1}{\underset{R_3 \to R_3 + 3R_1}{\smile}} \left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 8 & -8 & 0 \end{array}\right]$$

Page 315 Number 6 (continued 6)

Solution (continued). So the collection of all eigenvectors of $\lambda_2 = 2$ is

$$\boxed{\vec{v}_2 = s \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \text{ where } s \in \mathbb{R},\ s \neq 0.}$$

$\underline{\lambda_3 = 3.}$ As above, we consider $(A - 3\mathcal{I})\vec{v}_3 = \vec{0}$ and consider the augmented matrix

$$\left[\begin{array}{ccc|c} -3-(3) & 5 & -20 & 0 \\ 2 & 0-(3) & 8 & 0 \\ 2 & 1 & 7-(3) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -6 & 5 & -20 & 0 \\ 2 & -3 & 8 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right]$$

$$\underset{\sim}{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 2 & -3 & 8 & 0 \\ -6 & 5 & -20 & 0 \end{array}\right] \underset{R_3 \to R_3 + 3R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \left[\begin{array}{ccc|c} 2 & 1 & 4 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 8 & -8 & 0 \end{array}\right]$$

Page 315 Number 6 (continued 7)

Solution (continued).

$$\left[\begin{array}{rrr|r} 2 & 1 & 4 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 8 & -8 & 0 \end{array}\right] \underset{R_3 \to R_3/8}{\overset{R_2 \to R_2/(-4)}{\sim}} \left[\begin{array}{rrr|r} 2 & 1 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3 - R_2}{\overset{R_1 \to R_1 - R_2}{\sim}} \left[\begin{array}{rrr|r} 2 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_1/2}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 5/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{rcrcl} v_1 & & +(5/2)v_3 & = & 0 \\ & v_2 & -v_3 & = & 0 \\ & & 0 & = & 0 \end{array}$ or $\begin{array}{rcl} v_1 & = & -(5/2)v_3 \\ v_2 & = & v_3 \\ v_3 & = & v_3 \end{array}$ or, with $t = v_3/2$ as a free variable, $\begin{array}{rcr} v_1 & = & -5t \\ v_2 & = & 2t \\ v_3 & = & 2t \end{array}$.

Page 315 Number 6 (continued 7)

Solution (continued).

$$\left[\begin{array}{rrr|r} 2 & 1 & 4 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 8 & -8 & 0 \end{array}\right] \underset{R_3 \to R_3/8}{\overset{R_2 \to R_2/(-4)}{\sim}} \left[\begin{array}{rrr|r} 2 & 1 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3 - R_2}{\overset{R_1 \to R_1 - R_2}{\sim}} \left[\begin{array}{rrr|r} 2 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_1/2}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 5/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{rcrcl} v_1 & & + (5/2)v_3 & = & 0 \\ & v_2 & - v_3 & = & 0 \\ & & 0 & = & 0 \end{array}$ or $\begin{array}{rcr} v_1 & = & -(5/2)v_3 \\ v_2 & = & v_3 \\ v_3 & = & v_3 \end{array}$ or, with $t = v_3/2$ as a free variable, $\begin{array}{rcr} v_1 & = & -5t \\ v_2 & = & 2t \\ v_3 & = & 2t \end{array}$.

Page 315 Number 6 (continued 8)

Solution (continued). So the collection of all eigenvectors of $\lambda_3 = 3$ is

$$\vec{v}_3 = t \begin{bmatrix} -5 \\ 2 \\ 2 \end{bmatrix} \text{ where } t \in \mathbb{R},\ t \neq 0.$$

If we take $r = s = t = 1$ then we have the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = 3$ with corresponding eigenvalues $\vec{v}_1 = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -5 \\ 2 \\ 2 \end{bmatrix}$, respectively.

Page 315 Number 6 (continued 8)

Solution (continued). So the collection of all eigenvectors of $\lambda_3 = 3$ is

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Page 315 Number 6 (continued 8)

Solution (continued). So the collection of all eigenvectors of $\lambda_3 = 3$ is

$$\boxed{\vec{v}_3 = t\begin{bmatrix} -5 \\ 2 \\ 2 \end{bmatrix} \text{ where } t \in \mathbb{R},\ t \neq 0.}$$

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Page 315 Number 6 (continued 9)

Page 315 Number 6. Find the eigenvalues λ_i and corresponding eigenvectors $\vec{v}_i$ of $A = \begin{bmatrix} -3 & 5 & -20 \\ 2 & 0 & 8 \\ 2 & 1 & 7 \end{bmatrix}$. Find an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$.

Solution (continued). ... and $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ we have $AC = CD$. By Theorem 5.3, "Independence of Eigenvalues," we have that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent vectors and A is diagonalizable (notice that C is invertible by Theorem 1.16, "The Square Case, $m = n$"). So $D = C^{-1}AC$ where

$$C = \begin{bmatrix} -5 & -3 & -5 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

□

Page 315 Number 6 (continued 9)

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□

Page 315 Number 18

Page 315 Number 18. Prove that similar square matrices have the same eigenvalues with the same algebraic multiplicities.

Proof. (This is repetitious with Exercise 5.1.38.) Notice that

$$\begin{aligned} C^{-1}AC - \lambda\mathcal{I} &= C^{-1}AC - \lambda C^{-1}C \\ &= C^{-1}AC - C^{-1}(\lambda C) \text{ by Theorem 1.3.A(7),} \\ &\quad \text{“Scalars Pull Through”} \\ &= C^{-1}(AC - \lambda C) \text{ by Theorem 1.3.A(10),} \\ &\quad \text{“Distribution Law of Matrix Multiplication”} \\ &= C^{-1}(A - \lambda\mathcal{I})C \text{ by Theorem 1.3.A(10).} \end{aligned}$$

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Proof. (This is repetitious with Exercise 5.1.38.) Notice that

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Page 315 Number 18 (continued)

Proof (continued). Recall that $\det(C^{-1}) = 1/\det(C)$ by Exercise 4.2.31. So the characteristic polynomial for $C^{-1}AC$ is

$$\begin{aligned}
\det(C^{-1}AC - \lambda\mathcal{I}) &= \det(C^{-1}(A - \lambda\mathcal{I})C) \text{ as just shown} \\
&= \det(C^{-1})\det(A - \lambda\mathcal{I})\det(C) \text{ by Theorem 4.4,} \\
&\quad \text{“The Multiplicative Property”} \\
&= (1/\det(C))\det(A - \lambda\mathcal{I})\det(C) \\
&= \det(A - \lambda\mathcal{I}).
\end{aligned}$$

Page 315 Number 18 (continued)

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&= \det(A - \lambda\mathcal{I}).
\end{aligned}$$

Now $\det(A - \lambda\mathcal{I})$ is the characteristic polynomial of A, so A and $C^{-1}AC$ have the same characteristic polynomials. So these polynomials have the same roots with the same multiplicities (of course) and since the eigenvalues of a matrix are the roots of the characteristic polynomial, then A and $C^{-1}AC$ have the same eigenvalues with the same algebraic mulitplicities, as claimed. □

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$$\begin{aligned}\det(C^{-1}AC - \lambda\mathcal{I}) &= \det(C^{-1}(A - \lambda\mathcal{I})C) \text{ as just shown} \\ &= \det(C^{-1})\det(A - \lambda\mathcal{I})\det(C) \text{ by Theorem 4.4,} \\ &\quad \text{“The Multiplicative Property”} \\ &= (1/\det(C))\det(A - \lambda\mathcal{I})\det(C) \\ &= \det(A - \lambda\mathcal{I}).\end{aligned}$$

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Page 315 Number 10

Page 315 Number 10. Determine whether $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable.

Solution. First, we find the eigenvalues of A. Notice that $A - \lambda\mathcal{I} = \begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}$ is upper triangular, so by Example 4.2.4, $p(\lambda) = \det(A - \lambda\mathcal{I}) = (3-\lambda)^3$. So $\lambda = 3$ is the only eigenvalue of A and it is of algebraic multiplicity 3.

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$$\left[\begin{array}{ccc|c} 3-(3) & 1 & 0 & 0 \\ 0 & 3-(3) & 1 & 0 \\ 0 & 0 & 3-(3) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

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Page 315 Number 10 (continued)

Page 315 Number 10. Determine whether $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable.

Solution (continued). So we need
$$\begin{aligned} v_2 \quad &= 0 \\ v_3 &= 0 \\ 0 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} v_1 &= v_1 \\ v_2 &= 0 \\ v_3 &= 0 \end{aligned} \quad \text{or,}$$
with $r = v_1$ as a free variable, $\begin{aligned} v_1 &= r \\ v_2 &= 0 \\ v_3 &= 0 \end{aligned}$. So the collection of all eigenvectors of $\lambda = 3$ is $\vec{v} = r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ where $r \in \mathbb{R}$, $r \neq 0$. But then there can be only one vector in a set of linearly independent eigenvectors. That is, the dimension of E_λ is 1 and so $\lambda = 3$ is of geometric multiplicity 1. So, by Theorem 5.4, "A Criterion for Diagonalization,"

A is not diagonalizable. □

Page 315 Number 10 (continued)

Page 315 Number 10. Determine whether $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable.

Solution (continued). So we need $\begin{array}{rcl} v_2 & = & 0 \\ v_3 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcl} v_1 & = & v_1 \\ v_2 & = & 0 \\ v_3 & = & 0 \end{array}$ or,

with $r = v_1$ as a free variable, $\begin{array}{rcl} v_1 & = & r \\ v_2 & = & 0 \\ v_3 & = & 0 \end{array}$. So the collection of all eigenvectors of $\lambda = 3$ is $\vec{v} = r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ where $r \in \mathbb{R}$, $r \neq 0$. But then there can be only one vector in a set of linearly independent eigenvectors. That is, the dimension of E_λ is 1 and so $\lambda = 3$ is of geometric multiplicity 1. So, by Theorem 5.4, "A Criterion for Diagonalization,"

$\boxed{A \text{ is not diagonalizable.}}$ □

Page 316 Number 22

Page 316 Number 22. Let A and C be $n \times n$ matrices, and let C be invertible. Prove that, if $\vec{v}$ is an eigenvector of A with corresponding eigenvalue λ, then $C^{-1}\vec{v}$ is an eigenvector of $C^{-1}AC$ with corresponding eigenvalue λ. Prove that all eigenvectors of $C^{-1}AC$ are of the form $C^{-1}\vec{v}$, where $\vec{v}$ is an eigenvector of A.

Proof. If $\vec{v}$ is an eigenvector of A corresponding to eigenvalue λ then $A\vec{v} = \lambda\vec{v}$.

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Proof. If $\vec{v}$ is an eigenvector of A corresponding to eigenvalue λ then $A\vec{v} = \lambda\vec{v}$. So

$$\begin{aligned} (C^{-1}AC)(C^{-1}\vec{v}) &= C^{-1}A(CC^{-1})\vec{v} \text{ by Theorem 1.3.A(8),} \\ &\quad \text{“Associativity of Matrix Multiplication”} \\ &= C^{-1}A\mathcal{I}\vec{v} = C^{-1}A\vec{v} \\ &= C^{-1}(\lambda\vec{v}) = \lambda(C^{-1}\vec{v}) \text{ by Theorem 1.3.A(7),} \\ &\quad \text{“Scalars Pull Through”} \end{aligned}$$

and so λ is an eigenvalue of $C^{-1}AC$ with corresponding eigenvector $C^{-1}\vec{v}$, as claimed.

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&= C^{-1}(\lambda\vec{v}) = \lambda(C^{-1}\vec{v}) \text{ by Theorem 1.3.A(7),} \\
&\quad \text{“Scalars Pull Through”}
\end{aligned}$$

and so λ is an eigenvalue of $C^{-1}AC$ with corresponding eigenvector $C^{-1}\vec{v}$, as claimed.

Page 316 Number 22 (continued)

Page 316 Number 22. Let A and C be $n \times n$ matrices, and let C be invertible. Prove that, if $\vec{v}$ is an eigenvector of A with corresponding eigenvalue λ, then $C^{-1}\vec{v}$ is an eigenvector of $C^{-1}AC$ with corresponding eigenvalue λ. Prove that all eigenvectors of $C^{-1}AC$ are of the form $C^{-1}\vec{v}$, where $\vec{v}$ is an eigenvector of A.

Proof (continued). Now suppose $\vec{w}$ is an eigenvector of $C^{-1}AC$. Then $C^{-1}AC\vec{w} = \lambda\vec{w}$ for some $\lambda \in \mathbb{R}$. Then $C(C^{-1}AC\vec{w}) = C\lambda\vec{w}$ or $(CC^{-1})AC\vec{w} = \lambda C\vec{w}$ or $A(C\vec{w}) = \lambda(C\vec{w})$. So $\vec{v} = C\vec{w}$ is an eigenvector of A with corresponding eigenvalue λ. Then $\vec{w} = C^{-1}\vec{v}$ and so all eigenvectors of $C^{-1}AC$ are of the form $C^{-1}\vec{v}$ where $\vec{v}$ is an eigenvector of A, as claimed. □

Page 316 Number 22 (continued)

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Proof (continued). Now suppose $\vec{w}$ is an eigenvector of $C^{-1}AC$. Then $C^{-1}AC\vec{w} = \lambda\vec{w}$ for some $\lambda \in \mathbb{R}$. Then $C(C^{-1}AC\vec{w}) = C\lambda\vec{w}$ or $(CC^{-1})AC\vec{w} = \lambda C\vec{w}$ or $A(C\vec{w}) = \lambda(C\vec{w})$. So $\vec{v} = C\vec{w}$ is an eigenvector of A with corresponding eigenvalue λ. Then $\vec{w} = C^{-1}\vec{v}$ and so all eigenvectors of $C^{-1}AC$ are of the form $C^{-1}\vec{v}$ where $\vec{v}$ is an eigenvector of A, as claimed. □

Page 316 Number 24

Page 316 Number 24. Prove that if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct real eigenvalues of an $n \times n$ real matrix A and if B_i is a basis for the eigenspace E_{λ_i}, then the union of the bases B_i is an independent set of vectors in $\mathbb{R}^n$.

Proof. Let $B_i = \{\vec{b}^i_1, \vec{b}^i_2, \ldots, \vec{b}^i_{n_i}\}$ for $i = 1, 2, \ldots, k$, where n_i is the dimension of E_{λ_i}. Suppose

$$a^1_1 \vec{b}^1_1 + a^1_2 \vec{b}^1_2 + \cdots + a^1_{n_1} \vec{b}^1_{n_1} + a^2_1 \vec{b}^2_1 + a^2_2 \vec{b}^2_2 + \cdots + a^2_{n_2} \vec{b}^2_{n_2} + \cdots$$

$$+ a^k_1 \vec{b}^k_1 + a^k_2 \vec{b}^k_2 + \cdots + a^k_{n_k} \vec{b}^k_{n_k} = \vec{0}. \qquad (*)$$

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Proof. Let $B_i = \{\vec{b}_1^i, \vec{b}_2^i, \ldots, \vec{b}_{n_i}^i\}$ for $i = 1, 2, \ldots, k$, where n_i is the dimension of E_{λ_i}. Suppose

$$a_1^1 \vec{b}_1^1 + a_2^1 \vec{b}_2^1 + \cdots + a_{n_1}^1 \vec{b}_{n_1}^1 + a_1^2 \vec{b}_1^2 + a_2^2 \vec{b}_2^2 + \cdots + a_{n_2}^2 \vec{b}_{n_2}^2 + \cdots$$

$$+ a_1^k \vec{b}_1^k + a_2^k \vec{b}_2^k + \cdots + a_{n_k}^k \vec{b}_{n_k}^k = \vec{0}. \qquad (*)$$

If we let $\vec{w}_i = a_1^i \vec{b}_1^i + a_2^i \vec{b}_2^i + \cdots + a_{n_1}^i \vec{b}_{n_1}^i$ then $(*)$ gives $\vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_k = \vec{0}$. But $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent by Theorem 5.3, "Independence of Eigenvectors." This implies that each $\vec{w}_i$ must in fact be the zero vector, $\vec{w}_i = \vec{0}$ (or else some nonzero $\vec{w}_i$ is a linear combination of the other $\vec{w}_i$'s, contradicting the linear independence).

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Proof. Let $B_i = \{\vec{b}_1^i, \vec{b}_2^i, \ldots, \vec{b}_{n_i}^i\}$ for $i = 1, 2, \ldots, k$, where n_i is the dimension of E_{λ_i}. Suppose

$$a_1^1\vec{b}_1^1 + a_2^1\vec{b}_2^1 + \cdots + a_{n_1}^1\vec{b}_{n_1}^1 + a_1^2\vec{b}_1^2 + a_2^2\vec{b}_2^2 + \cdots + a_{n_2}^2\vec{b}_{n_2}^2 + \cdots$$

$$+a_1^k\vec{b}_1^k + a_2^k\vec{b}_2^k + \cdots + a_{n_k}^k\vec{b}_{n_k}^k = \vec{0}. \qquad (*)$$

If we let $\vec{w}_i = a_1^i\vec{b}_1^i + a_2^i\vec{b}_2^i + \cdots + a_{n_1}^i\vec{b}_{n_1}^i$ then $(*)$ gives $\vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_k = \vec{0}$. But $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent by Theorem 5.3, "Independence of Eigenvectors." This implies that each $\vec{w}_i$ must in fact be the zero vector, $\vec{w}_i = \vec{0}$ (or else some nonzero $\vec{w}_i$ is a linear combination of the other $\vec{w}_i$'s, contradicting the linear independence).

Page 316 Number 24 (continued)

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Proof (continued). But then $\vec{w}_i = a_1^i \vec{b}_1^i + a_2^i \vec{b}_2^i + \cdots + a_{n_1}^i \vec{b}_{n_1}^i = \vec{0}$ and since the $\vec{b}_j^i$'s are a basis for E_{λ_i} then the $\vec{b}_j^i$ are linear independent for a given i and so each $a_j^i = 0$ for given i. Since this holds for all $i = 1, 2, \ldots, k$ then all $a_j^i = 0$ and so we see from $(*)$ that $\{\vec{b}_1^1, \vec{b}_2^1, \ldots, \vec{b}_{n_k}^k\} = \cup_{i=1}^k B_i$ is a linearly independent set. □

Page 316 Number 26

Page 316 Number 26. Prove that the set $\{e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}\}$, where the λ_i are distinct, is independent in the vector space D_∞ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ and having derivatives of all orders (see Note 3.2.A).

Proof. We know that differentiation D is a linear transformation mapping D_∞ into D_∞ (see Example 3.4.1). Now $D(e^{\lambda_i x}) = \frac{d}{dx}[e^{\lambda_i x}] = \lambda_i e^{\lambda_i x}$, so $e^{\lambda_i x}$ is an eigenvector of D with corresponding eigenvalue λ_i.

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