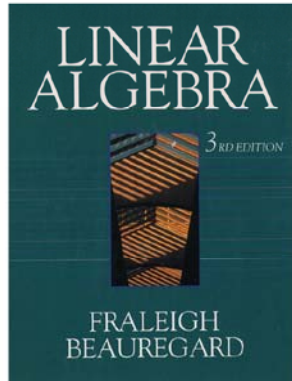# Linear Algebra

Chapter 5: Eigenvalues and Eigenvectors
Section 5.3. Two Applications—Proofs of Theorems

Page 325 Number 4

Page 325 Number 4. Let the sequence $a_0, a_1, a_2, \ldots$ be given by $a_0 = 0, 1_1 = 1$, and $a_k = (1/2)a_{k-1} + (3/16)a_{k-2}$ for $k \geq 2$. Find the matrix A that can be used to generate this sequence, classify this process as stable, neutrally stable, or unstable, and compute a_k for

$$\vec{x} = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution. Since $\vec{x} = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$ and $A\vec{x} = \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a_1 + \frac{3}{16}a_0 \\ a_1 \end{bmatrix}$ then we

need $A = \begin{bmatrix} 1/2 & 3/16 \\ 1 & 0 \end{bmatrix}$.

To address stability, we need to find the eigenvalues of A. We need

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} \frac{1}{2} - \lambda & \frac{3}{16} \\ 1 & -\lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda\right)(-\lambda) - \left(\frac{3}{16}\right)(1)$$

$$= \lambda^2 - \frac{1}{2}\lambda - \frac{3}{16} = \left(\lambda + \frac{1}{4}\right)\left(\lambda - \frac{3}{4}\right) = 0.$$

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Solution (continued). So the eigenvalues of A are $\lambda_1 = -1/4$ and $\lambda_2 = 3/4$. Since $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then the process is stable.

To compute $A^k\vec{x}$, we use Equation (1) and we need the eigenvectors of A. For $\lambda_1 = -1/4$ with eigenvector $\vec{v}_1 = [v_1, v_2]^T$ we need $A\vec{v}_1 = \lambda\vec{v}_1$ or $(A - \lambda_1\mathcal{I})\vec{v}_1 = \vec{0}$ so we consider the augmented matrix

$$[A - \lambda_1\mathcal{I} \mid \vec{0}] = \begin{bmatrix} 3/4 & 3/16 & 0 \\ 1 & 1/4 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1/4 & 0 \\ 3/4 & 3/16 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - (3/4)R_1} \begin{bmatrix} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we need $\begin{aligned} v_1 + (1/4)v_2 &= 0 \\ 0 &= 0 \end{aligned}$ or $\begin{aligned} v_1 &= (-1/4)v_2 \\ v_2 &= v_2 \end{aligned}$ or, with

$r = v_2/4$ as a free variable, $\begin{aligned} v_1 &= -r \\ v_2 &= 4r \end{aligned}$ where $r \in \mathbb{R}, r \neq 0$.

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Solution (continued). We choose $r = 1$ and $\vec{v}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

Similarly for $\lambda_2 = 3/4$ with eigenvector $\vec{v}_2 = [v_1, v_2]^T$ we consider

$$[A - \lambda_2\mathcal{I} \mid \vec{0}] = \begin{bmatrix} -1/4 & 3/16 & 0 \\ 1 & -3/4 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -3/4 & 0 \\ -1/4 & 3/16 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + (1/4)R_1} \begin{bmatrix} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we need $\begin{aligned} v_1 - (3/4)v_2 &= 0 \\ 0 &= 0 \end{aligned}$ or $\begin{aligned} v_1 &= (3/4)v_2 \\ v_2 &= v_2 \end{aligned}$ or, with

$s = v_2/4$ as a free variable, $\begin{aligned} v_1 &= 3s \\ v_2 &= 4s \end{aligned}$ where $s \in \mathbb{R}, s \neq 0$. We choose

$s = 1$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

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Solution (continued). To use Equation (1), we need $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, $\vec{x} = d_1\vec{v}_1 + d_2\vec{d}_2 = C\vec{d}$, or $\vec{d} = C^{-1}\vec{x}$ where the columns of C are eigenvectors $\vec{v}_1$ and $\vec{v}_2$. So we find C^{-1} as follows:

$$[C \mid \mathcal{I}] = \begin{bmatrix} -1 & 3 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -1 & 3 & 1 & 0 \\ 0 & 16 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2/16]{R_1 \to -R_1} \begin{bmatrix} 1 & -3 & -1 & 0 \\ 0 & 1 & 1/4 & 1/16 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 0 & -1/4 & 3/16 \\ 0 & 1 & 1/4 & 1/16 \end{bmatrix}$$

and so $C^{-1} = \begin{bmatrix} -1/4 & 3/16 \\ 1/4 & 1/16 \end{bmatrix}$. Therefore

$$\vec{d} = C^{-1}\vec{x} = \begin{bmatrix} -1/4 & 3/16 \\ 1/4 & 1/16 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

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Solution (continued). So $\vec{x} = d_1\vec{v}_1 + d_2\vec{v}_2 = \frac{-1}{4}\begin{bmatrix} -1 \\ 4 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 3 \\ r \end{bmatrix}$.

Equation (1) then gives

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = A^k\vec{x} = d_1\lambda_1^k\vec{v}_1 + d_2\lambda_2^k\vec{v}_2$$

$$= \left(\frac{-1}{4}\right)\left(\frac{-1}{4}\right)^k\begin{bmatrix} -1 \\ 4 \end{bmatrix} + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^k\begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

or

$$a_k = \left(\frac{-1}{4}\right)\left(\frac{-1}{4}\right)^k(4) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^k(4) = \boxed{\left(\frac{3}{4}\right)^k - \left(\frac{-1}{4}\right)^k}.$$

$\square$

Page 325 Number 10

Page 325 Number 10. Find the general solution to the system:

$$\begin{aligned} x_1' &= 6x_1 + 3x_2 - 2x_3 \\ x_2' &= -2x_1 - x_2 + 2x_3 \\ x_3' &= 16x_1 + 8x_2 - 7x_3. \end{aligned}$$

Solution. The coefficient matrix $A = \begin{bmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{bmatrix}$ is not diagonal, so we attempt to diagonalize it. First we need the eigenvalues, so consider $\det(A - \lambda\mathcal{I}) = 0$:

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} 6-\lambda & 3 & -3 \\ -2 & -1-\lambda & 2 \\ 16 & 8 & -7-\lambda \end{vmatrix} = \cdots$$

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Solution (continued). $\det(A - \lambda\mathcal{I}) =$

$$= (6-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 8 & -7-\lambda \end{vmatrix} - (3)\begin{vmatrix} -2 & 2 \\ 16 & -7-\lambda \end{vmatrix} + (-3)\begin{vmatrix} -2 & -1-\lambda \\ 16 & 8 \end{vmatrix}$$

$$= (6-\lambda)((-1-\lambda)(-7-\lambda) - (2)(8)) - 3((-2)(-7-\lambda) - (2)(16))$$

$$-3((-2)(8) - (-1-\lambda)(16)) = (6-\lambda)(\lambda^2 + 8\lambda - 9) - 3(2\lambda - 18) - 2(16\lambda)$$

$$= 6\lambda^2 + 48\lambda - 54 - \lambda^3 - 8\lambda^2 + 9\lambda - 6\lambda + 54 - 48\lambda = -\lambda^3 - 2\lambda^2 + 3\lambda$$

$$= -\lambda(\lambda^2 + 2\lambda - 3) = -\lambda(\lambda + 3)(\lambda - 1).$$

So the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 0$, and $\lambda_3 = 1$. Now we find corresponding eigenvectors.

For eigenvalue $\lambda_1 = -3$ with corresponding eigenvector $\vec{v}_1 = [v_1, v_2, v_3]^T$ we need $A\vec{v}_1 = \lambda_1\vec{v}_1$ or $(A - \lambda_1\mathcal{I})\vec{v}_1 = \vec{0}$. So we consider the augmented matrix...

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Solution (continued).

$$[A - \lambda_1 \mathcal{I} \mid \vec{0}] = \begin{bmatrix} 9 & 3 & -3 & | & 0 \\ -2 & 2 & 2 & | & 0 \\ 16 & 8 & -4 & | & 0 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2/(-2) \\ R_3 \to R_3/4}]{R_1 \to R_1/3} \begin{bmatrix} 3 & 1 & -1 & | & 0 \\ 1 & -1 & -1 & | & 0 \\ 4 & 2 & -1 & | & 0 \end{bmatrix}$$

$$\xrightarrow[]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & -1 & | & 0 \\ 3 & 1 & -1 & | & 0 \\ 4 & 2 & -1 & | & 0 \end{bmatrix} \xrightarrow[\substack{R_4 \to R_4 - 4R_1}]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & -1 & -1 & | & 0 \\ 0 & 4 & 2 & | & 0 \\ 0 & 6 & 3 & | & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3 \to R_3/3}]{R_2 \to R_2/4} \begin{bmatrix} 1 & -1 & -1 & | & 0 \\ 0 & 1 & 1/2 & | & 0 \\ 0 & 2 & 1 & | & 0 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 2R_2}]{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & -1/2 & | & 0 \\ 0 & 1 & 1/2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

and we need

$$\begin{array}{rcl} v_1 & -(1/2)v_3 = 0 \\ v_2 & +(1/2)v_3 = 0 \\ 0 & = 0 \end{array} \quad \text{or} \quad \begin{array}{rcl} v_1 & = & (1/2)v_3 \\ v_2 & = & -(1/2)v_3 \\ v_3 & = & v_3 \end{array} \quad \text{or, } \ldots$$

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Solution (continued). ... with $r = v_3/2$ as a free variable,
$$\begin{array}{rcl} v_1 & = & r \\ v_2 & = & -r \\ v_3 & = & 2r \end{array}$$

where $r \in \mathbb{R}$, $r \neq 0$. We choose $r = 1$ and $\vec{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Similarly, for eigenvalue $\lambda_2 = 0$ with corresponding eigenvector $\vec{v}_2 = [v_1, v_2, v_3]^T$ we consider

$$[A - \lambda_2 \mathcal{I} \mid \vec{0}] = \begin{bmatrix} 6 & 3 & -3 & | & 0 \\ -2 & -1 & 2 & | & 0 \\ 16 & 8 & -7 & | & 0 \end{bmatrix} \xrightarrow[]{R_1 \leftrightarrow R_2} \begin{bmatrix} -2 & -1 & 2 & | & 0 \\ 6 & 3 & -3 & | & 0 \\ 16 & 8 & -7 & | & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3 \to R_3 + 8R_1}]{R_2 \to R_2 + 3R_1} \begin{bmatrix} -2 & -1 & 2 & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 9 & | & 0 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 3R_2}]{R_1 \to R_1/(-2)} \begin{bmatrix} 1 & 1/2 & -1 & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

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Solution (continued). ...

$$\xrightarrow[]{R_2 \to R_2/3} \begin{bmatrix} 1 & 1/2 & -1 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow[]{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 1/2 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

and we need

$$\begin{array}{rcl} v_1 + (1/2)v_2 & = & 0 \\ v_3 & = & 0 \\ 0 & = & 0 \end{array} \quad \text{or} \quad \begin{array}{rcl} v_1 & = & -(1/2)v_2 \\ v_2 & = & v_2 \\ v_3 & = & 0 \end{array} \quad \text{or, with}$$

$s = v_2/2$ as a free variable,
$$\begin{array}{rcl} v_1 & = & -s \\ v_2 & = & 2s \\ v_3 & = & 0 \end{array} \quad \text{where } s \in \mathbb{R},\ s \neq 0. \text{ We choose}$$

$s = 1$ and $\vec{v}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

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Solution (continued). Similarly, for eigenvalue $\lambda_3 = 1$ with corresponding eigenvector $\vec{v}_3 = [v_1, v_2, v_3]^T$ we consider

$$[A - \lambda_3 \mathcal{I} \mid \vec{0}] = \begin{bmatrix} 5 & 3 & -3 & | & 0 \\ -2 & -2 & 2 & | & 0 \\ 16 & 8 & -8 & | & 0 \end{bmatrix} \xrightarrow[]{R_1 \leftrightarrow R_2} \begin{bmatrix} -2 & -2 & 2 & | & 0 \\ 5 & 3 & -3 & | & 0 \\ 16 & 8 & -8 & | & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3 \to R_3/8}]{R_1 \to R_1/(-2)} \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 5 & 3 & -3 & | & 0 \\ 2 & 1 & -1 & | & 0 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 2R_1}]{R_2 \to R_2 - 5R_1} \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 2 & | & 0 \\ 0 & -1 & 1 & | & 0 \end{bmatrix}$$

$$\xrightarrow[]{R_3 \to R_3 - (1/2)R_2} \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow[]{R_2 \to R_2/(-2)} \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\xrightarrow[]{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

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Solution (continued). ...and we need

$$
\begin{aligned}
v_1 &= 0 \\
v_2 - v_3 &= 0 \quad \text{or} \\
0 &= 0
\end{aligned}
$$

$$
\begin{aligned}
v_1 &= 0 \\
v_2 &= v_3 \\
v_3 &= v_3
\end{aligned}
\quad \text{or, with } t = v_3 \text{ as a free variable,} \quad
\begin{aligned}
v_1 &= 0 \\
v_2 &= t \quad \text{for } t \in \mathbb{R}, \\
v_3 &= t
\end{aligned}
$$

$t \neq 0$. We choose $t = 1$ and $\vec{v}_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. With C the matrix

whose columns are $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ we have $C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$.

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Solution (continued). In the notation introduced above, the solution to

$$
\vec{y}' = D\vec{y} \text{ is } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} k_1 e^{\lambda_1 t} \\ k_2 e^{\lambda_2 t} \\ k_3 e^{\lambda_3 t} \end{bmatrix} = \begin{bmatrix} k_1 e^{-3t} \\ k_2 \\ k_3 e^{t} \end{bmatrix} \text{ and the solution to}
$$

the original system is

$$
\vec{x} = C\vec{y} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 e^{-3t} \\ k_2 \\ k_3 e^{t} \end{bmatrix}
$$

$$
= \begin{bmatrix} k_1 e^{-3t} & - & k_2 & & \\ -k_1 e^{-3t} & + & 2k_2 & + & k_3 e^{t} \\ 2e^{-3t} & & & + & k_3 e^{t} \end{bmatrix}.
$$

$\square$