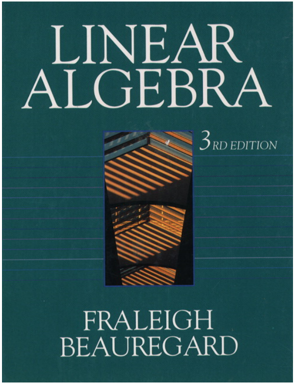
Linear Algebra

Chapter 5: Eigenvalues and Eigenvectors

Section 5.3. Two Applications—Proofs of Theorems



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Page 325 Number 4. Let the sequence $a_0, a_1, a_2, \ldots$ be given by $a_0 = 0, 1_1 = 1$, and $a_k = (1/2)a_{k-1} + (3/16)a_{k-2}$ for $k \geq 2$. Find the matrix A that can be used to generate this sequence, classify this process as stable, neutrally stable, or unstable, and compute a_k for $\vec{x} = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution. Since $\vec{x} = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$ and $A\vec{x} = \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a_1 + \frac{3}{16}a_0 \\ a_1 \end{bmatrix}$ then we need $\boxed{A = \begin{bmatrix} 1/2 & 3/16 \\ 1 & 0 \end{bmatrix}.}$

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To address stability, we need to find the eigenvalues of A. We need

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} \frac{1}{2} - \lambda & \frac{3}{16} \\ 1 & -\lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda\right)(-\lambda) - \left(\frac{3}{16}\right)(1)$$

$$= \lambda^2 - \frac{1}{2}\lambda - \frac{3}{16} = \left(\lambda + \frac{1}{4}\right)\left(\lambda - \frac{3}{4}\right) = 0.$$

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Solution (continued). So the eigenvalues of A are $\lambda_1 = -1/4$ and $\lambda_2 = 3/4$. Since $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then the process is $\boxed{\text{stable.}}$

To compute $A^k \vec{x}$, we use Equation (1) and we need the eigenvectors of A. For $\lambda_1 = -1/4$ with eigenvector $\vec{v}_1 = [v_1, v_2]^T$ we need $A\vec{v}_1 = \lambda \vec{v}_1$ or $(A - \lambda_1 \mathcal{I})\vec{v}_1 = \vec{0}$ so we consider the augmented matrix

$$[A - \lambda_1 \mathcal{I} \mid \vec{0}] = \left[\begin{array}{cc|c} 3/4 & 3/16 & 0 \\ 1 & 1/4 & 0 \end{array}\right] \underset{\sim}{\overset{R_1 \leftrightarrow R_2}{}} \left[\begin{array}{cc|c} 1 & 1/4 & 0 \\ 3/4 & 3/16 & 0 \end{array}\right]$$

$$\underset{\sim}{\overset{R_2 \to R_2 - (3/4)R_1}{}} \left[\begin{array}{cc|c} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

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and we need $\begin{array}{rcl} v_1 + (1/4)v_2 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcr} v_1 & = & (-1/4)v_2 \\ v_2 & = & v_2 \end{array}$ or, with $r = v_2/4$ as a free variable, $\begin{array}{rcr} v_1 & = & -r \\ v_2 & = & 4r \end{array}$ where $r \in \mathbb{R}$, $r \neq 0$.

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Solution (continued). We choose $r = 1$ and $\vec{v}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

Similarly for $\lambda_2 = 3/4$ with eigenvector $\vec{v}_2 = [v_1, v_2]^T$ we consider

$$[A - \lambda_2 \mathcal{I} \mid \vec{0}] = \left[\begin{array}{cc|c} -1/4 & 3/16 & 0 \\ 1 & -3/4 & 0 \end{array}\right] \underset{\sim}{\overset{R_1 \leftrightarrow R_2}{}} \left[\begin{array}{cc|c} 1 & -3/4 & 0 \\ -1/4 & 3/16 & 0 \end{array}\right]$$

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Solution (continued). To use Equation (1), we need $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, $\vec{x} = d_1\vec{v}_1 + d_2\vec{d}_2 = C\vec{d}$, or $\vec{d} = C^{-1}\vec{x}$ where the columns of C are eigenvectors $\vec{v}_1$ and $\vec{v}_2$. So we find C^{-1} as follows:

$$[C \mid \mathcal{I}] = \left[\begin{array}{rr|rr} -1 & 3 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{array}\right] \underbrace{}_{R_2 \to R_2 + R_1} \left[\begin{array}{rr|rr} -1 & 3 & 1 & 0 \\ 0 & 16 & 4 & 1 \end{array}\right]$$

$$\underbrace{}_{\substack{R_1 \to -R_1 \\ R_2 \to R_2/16}} \left[\begin{array}{rr|rr} 1 & -3 & -1 & 0 \\ 0 & 1 & 1/4 & 1/16 \end{array}\right] \underbrace{}_{R_2 \to R_2 + 3R_1} \left[\begin{array}{rr|rr} 1 & 0 & -1/4 & 3/16 \\ 0 & 1 & 1/4 & 1/16 \end{array}\right]$$

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Solution (continued). So $\vec{x} = d_1\vec{v}_1 + d_2\vec{v}_2 = \frac{-1}{4}\begin{bmatrix} -1 \\ 4 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 3 \\ r \end{bmatrix}$.

Equation (1) then gives

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = A^k\vec{x} = d_1\lambda_1^k\vec{v}_1 + d_2\lambda_2^k\vec{v}_2$$

$$= \left(\frac{-1}{4}\right)\left(\frac{-1}{4}\right)^k\begin{bmatrix} -1 \\ 4 \end{bmatrix} + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^k\begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

or

$$a_k = \left(\frac{-1}{4}\right)\left(\frac{-1}{4}\right)^k(4) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^k(4) = \boxed{\left(\frac{3}{4}\right)^k - \left(\frac{-1}{4}\right)^k.}$$

□

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□

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$$\begin{array}{rcrcrcr} x_1' & = & 6x_1 & + & 3x_2 & - & 2x_3 \\ x_2' & = & -2x_1 & - & x_2 & + & 2x_3 \\ x_3' & = & 16x_1 & + & 8x_2 & - & 7x_3. \end{array}$$

Solution. The coefficient matrix $A = \begin{bmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{bmatrix}$ is not diagonal, so we attempt to diagonalize it.

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$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} 6-\lambda & 3 & -3 \\ -2 & -1-\lambda & 2 \\ 16 & 8 & -7-\lambda \end{vmatrix} = \cdots$$

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Solution (continued). $\det(A - \lambda\mathcal{I}) =$

$$= (6-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 8 & -7-\lambda \end{vmatrix} - (3)\begin{vmatrix} -2 & 2 \\ 16 & -7-\lambda \end{vmatrix} + (-3)\begin{vmatrix} -2 & -1-\lambda \\ 16 & 8 \end{vmatrix}$$

$$= (6-\lambda)((-1-\lambda)(-7-\lambda) - (2)(8)) - 3((-2)(-7-\lambda) - (2)(16))$$

$$-3((-2)(8) - (-1-\lambda)(16)) = (6-\lambda)(\lambda^2 + 8\lambda - 9) - 3(2\lambda - 18) - 2(16\lambda)$$

$$= 6\lambda^2 + 48\lambda - 54 - \lambda^3 - 8\lambda^2 + 9\lambda - 6\lambda + 54 - 48\lambda = -\lambda^3 - 2\lambda^2 + 3\lambda$$

$$= -\lambda(\lambda^2 + 2\lambda - 3) = -\lambda(\lambda + 3)(\lambda - 1).$$

So the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 0$, and $\lambda_3 = 1$. Now we find corresponding eigenvectors.

For eigenvalue $\lambda_1 = -3$ with corresponding eigenvector $\vec{v}_1 = [v_1, v_2, v_3]^T$ we need $A\vec{v}_1 = \lambda_1\vec{v}_1$ or $(A - \lambda_1\mathcal{I})\vec{v}_1 = \vec{0}$. So we consider the augmented matrix. . .

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Solution (continued). $\det(A - \lambda\mathcal{I}) =$

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So the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 0$, and $\lambda_3 = 1$. Now we find corresponding eigenvectors.

For eigenvalue $\lambda_1 = -3$ with corresponding eigenvector $\vec{v}_1 = [v_1, v_2, v_3]^T$ we need $A\vec{v}_1 = \lambda_1\vec{v}_1$ or $(A - \lambda_1\mathcal{I})\vec{v}_1 = \vec{0}$. So we consider the augmented matrix. . .

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Solution (continued).

$$[A - \lambda_1 \mathcal{I} \mid \vec{0}] = \left[\begin{array}{rrr|r} 9 & 3 & -3 & 0 \\ -2 & 2 & 2 & 0 \\ 16 & 8 & -4 & 0 \end{array}\right] \underset{R_3 \to R_3/4}{\overset{R_1 \to R_1/3}{\widetilde{R_2 \to R_2/(-2)}}} \left[\begin{array}{rrr|r} 3 & 1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 4 & 2 & -1 & 0 \end{array}\right]$$

$$\widetilde{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 3 & 1 & -1 & 0 \\ 4 & 2 & -1 & 0 \end{array}\right] \underset{R_4 \to R_4 - 4R_1}{\widetilde{R_2 \to R_2 - 3R_1}} \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 6 & 3 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3/3}{\widetilde{R_2 \to R_2/4}} \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \underset{R_3 \to R_3 - 2R_2}{\widetilde{R_1 \to R_1 + R_2}} \left[\begin{array}{rrr|r} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and we need $\begin{array}{rrrl} v_1 & -(1/2)v_3 & = & 0 \\ v_2 & +(1/2)v_3 & = & 0 \\ & 0 & = & 0 \end{array}$ or $\begin{array}{rcr} v_1 & = & (1/2)v_3 \\ v_2 & = & -(1/2)v_3 \\ v_3 & = & v_3 \end{array}$ or, ...

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$$[A-\lambda_1\mathcal{I} \mid \vec{0}] = \left[\begin{array}{rrr|r} 9 & 3 & -3 & 0 \\ -2 & 2 & 2 & 0 \\ 16 & 8 & -4 & 0 \end{array}\right] \underset{R_3 \to R_3/4}{\underset{\sim}{\overset{R_1 \to R_1/3}{R_2 \to R_2/(-2)}}} \left[\begin{array}{rrr|r} 3 & 1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 4 & 2 & -1 & 0 \end{array}\right]$$

$$\underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 3 & 1 & -1 & 0 \\ 4 & 2 & -1 & 0 \end{array}\right] \underset{R_4 \to R_4 - 4R_1}{\underset{\sim}{R_2 \to R_2 - 3R_1}} \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 6 & 3 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3/3}{\underset{\sim}{R_2 \to R_2/4}} \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \underset{R_3 \to R_3 - 2R_2}{\underset{\sim}{R_1 \to R_1 + R_2}} \left[\begin{array}{rrr|r} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and we need
$$\begin{array}{rrrcl} v_1 & & -(1/2)v_3 & = & 0 \\ & v_2 & +(1/2)v_3 & = & 0 \\ & & 0 & = & 0 \end{array} \text{ or } \begin{array}{rcr} v_1 & = & (1/2)v_3 \\ v_2 & = & -(1/2)v_3 \\ v_3 & = & v_3 \end{array} \text{ or, } \ldots$$

Page 325 Number 10 (continued 3)

Solution (continued). . . . with $r = v_3/2$ as a free variable, $\begin{array}{rcr} v_1 & = & r \\ v_2 & = & -r \\ v_3 & = & 2r \end{array}$

where $r \in \mathbb{R}$, $r \neq 0$. We choose $r = 1$ and $\vec{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Similarly, for eigenvalue $\lambda_2 = 0$ with corresponding eigenvector $\vec{v}_2 = [v_1, v_2, v_3]^T$ we consider

$$[A - \lambda_2 \mathcal{I} \mid \vec{0}] = \left[\begin{array}{rrr|r} 6 & 3 & -3 & 0 \\ -2 & -1 & 2 & 0 \\ 16 & 8 & -7 & 0 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} -2 & -1 & 2 & 0 \\ 6 & 3 & -3 & 0 \\ 16 & 8 & -7 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 \to R_2 + 3R_1 \\ \sim \\ R_3 \to R_3 + 8R_1 \end{array} \left[\begin{array}{rrr|r} -2 & -1 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 9 & 0 \end{array}\right] \begin{array}{c} R_1 \to R_1/(-2) \\ \sim \\ R_3 \to R_3 - 3R_2 \end{array} \left[\begin{array}{rrr|r} 1 & 1/2 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Page 325 Number 10 (continued 3)

Solution (continued). ...with $r = v_3/2$ as a free variable, $\begin{array}{rcr} v_1 & = & r \\ v_2 & = & -r \\ v_3 & = & 2r \end{array}$

where $r \in \mathbb{R}$, $r \neq 0$. We choose $r = 1$ and $\vec{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Similarly, for eigenvalue $\lambda_2 = 0$ with corresponding eigenvector $\vec{v}_2 = [v_1, v_2, v_3]^T$ we consider

$$[A - \lambda_2 \mathcal{I} \mid \vec{0}] = \left[\begin{array}{rrr|r} 6 & 3 & -3 & 0 \\ -2 & -1 & 2 & 0 \\ 16 & 8 & -7 & 0 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} -2 & -1 & 2 & 0 \\ 6 & 3 & -3 & 0 \\ 16 & 8 & -7 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3 + 8R_1}{\overset{R_2 \to R_2 + 3R_1}{\sim}} \left[\begin{array}{rrr|r} -2 & -1 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 9 & 0 \end{array}\right] \underset{R_3 \to R_3 - 3R_2}{\overset{R_1 \to R_1/(-2)}{\sim}} \left[\begin{array}{rrr|r} 1 & 1/2 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Page 325 Number 10 (continued 4)

Solution (continued). ...

$$\underset{R_2 \to R_2/3}{\sim} \left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_1 \to R_1 + R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and we need $\begin{array}{rcl} v_1 + (1/2)v_2 & = & 0 \\ v_3 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcr} v_1 & = & -(1/2)v_2 \\ v_2 & = & v_2 \\ v_3 & = & 0 \end{array}$ or, with

$s = v_2/2$ as a free variable, $\begin{array}{rcr} v_1 & = & -s \\ v_2 & = & 2s \\ v_3 & = & 0 \end{array}$ where $s \in \mathbb{R}$, $s \neq 0$. We choose

$s = 1$ and $\vec{v}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

Page 325 Number 10 (continued 4)

Solution (continued). ...

$$\underset{R_2 \to R_2/3}{\sim} \left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_1 \to R_1 + R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and we need $\begin{array}{rcl} v_1 + (1/2)v_2 & = & 0 \\ v_3 & = & 0 \\ 0 & = & 0 \end{array}$ or $\begin{array}{rcr} v_1 & = & -(1/2)v_2 \\ v_2 & = & v_2 \\ v_3 & = & 0 \end{array}$ or, with

$s = v_2/2$ as a free variable, $\begin{array}{rcr} v_1 & = & -s \\ v_2 & = & 2s \\ v_3 & = & 0 \end{array}$ where $s \in \mathbb{R}$, $s \neq 0$. We choose

$s = 1$ and $\vec{v}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

Page 325 Number 10 (continued 5)

Solution (continued). Similarly, for eigenvalue $\lambda_3 = 1$ with corresponding eigenvector $\vec{v}_3 = [v_1, v_2, v_3]^T$ we consider

$$[A - \lambda_3 \mathcal{I} \mid \vec{0}] = \left[\begin{array}{rrr|r} 5 & 3 & -3 & 0 \\ -2 & -2 & 2 & 0 \\ 16 & 8 & -8 & 0 \end{array}\right] \underset{\sim}{\overset{R_1 \leftrightarrow R_2}{}} \left[\begin{array}{rrr|r} -2 & -2 & 2 & 0 \\ 5 & 3 & -3 & 0 \\ 16 & 8 & -8 & 0 \end{array}\right]$$

$$\underset{R_3 \to R_3/8}{\overset{R_1 \to R_1/(-2)}{\sim}} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 5 & 3 & -3 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right] \underset{R_3 \to R_3 - 2R_1}{\overset{R_2 \to R_2 - 5R_1}{\sim}} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{\overset{R_3 \to R_3 - (1/2)R_2}{}} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{\overset{R_2 \to R_2/(-2)}{}} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{\sim}{\overset{R_1 \to R_1 - R_2}{}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Page 325 Number 10 (continued 6)

Solution (continued). . . . and we need $\begin{array}{rcl} v_1 & = & 0 \\ v_2 - v_3 & = & 0 \\ 0 & = & 0 \end{array}$ or

$\begin{array}{rcl} v_1 & = & 0 \\ v_2 & = & v_3 \\ v_3 & = & v_3 \end{array}$ or, with $t = v_3$ as a free variable, $\begin{array}{rcl} v_1 & = & 0 \\ v_2 & = & t \\ v_3 & = & t \end{array}$ for $t \in \mathbb{R}$,

$t \neq 0$. We choose $t = 1$ and $\vec{v}_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. With C the matrix whose columns are $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ we have $\boxed{C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}}$.

Page 325 Number 10 (continued 6)

Solution (continued). . . . and we need $\begin{aligned} v_1 \quad\quad &= 0 \\ v_2 - v_3 &= 0 \\ 0 &= 0 \end{aligned}$ or

$\begin{aligned} v_1 &= 0 \\ v_2 &= v_3 \\ v_3 &= v_3 \end{aligned}$ or, with $t = v_3$ as a free variable, $\begin{aligned} v_1 &= 0 \\ v_2 &= t \\ v_3 &= t \end{aligned}$ for $t \in \mathbb{R}$,

$t \neq 0$. We choose $t = 1$ and $\vec{v}_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. With C the matrix whose columns are $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ we have $\boxed{C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}}$.

Page 325 Number 10 (continued 7)

Solution (continued). In the notation introduced above, the solution to $\vec{y}\,' = D\vec{y}$ is $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} k_1 e^{\lambda_1 t} \\ k_2 e^{\lambda_2 t} \\ k_3 e^{\lambda_3 t} \end{bmatrix} = \begin{bmatrix} k_1 e^{-3t} \\ k_2 \\ k_3 e^{t} \end{bmatrix}$ and the solution to the original system is

$$\vec{x} = C\vec{y} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 e^{-3t} \\ k_2 \\ k_3 e^{t} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} k_1 e^{-3t} & - & k_2 & & \\ -k_1 e^{-3t} & + & 2k_2 & + & k_3 e^{t} \\ 2e^{-3t} & & & + & k_3 e^{t} \end{bmatrix}}.$$

□

Page 325 Number 10 (continued 7)

Solution (continued). In the notation introduced above, the solution to $\vec{y}\,' = D\vec{y}$ is $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} k_1 e^{\lambda_1 t} \\ k_2 e^{\lambda_2 t} \\ k_3 e^{\lambda_3 t} \end{bmatrix} = \begin{bmatrix} k_1 e^{-3t} \\ k_2 \\ k_3 e^{t} \end{bmatrix}$ and the solution to the original system is

$$\vec{x} = C\vec{y} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 e^{-3t} \\ k_2 \\ k_3 e^{t} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} k_1 e^{-3t} & - & k_2 & & \\ -k_1 e^{-3t} & + & 2k_2 & + & k_3 e^{t} \\ 2e^{-3t} & & & + & k_3 e^{t} \end{bmatrix}}.$$

□