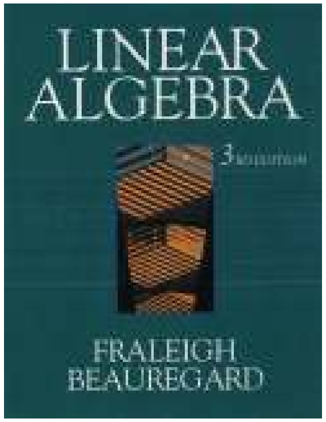
Linear Algebra

Chapter 6: Orthogonality

Section 6.2. The Gram-Schmidt Process—Proofs of Theorems



Theorem 6.2

Theorem 6.2. Orthogonal Bases.
Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be an orthogonal set of nonzero vectors in $\mathbb{R}^n$. Then this set is independent and consequently is a basis for the subspace $\text{sp}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$.

Proof. Let j be an integer between 2 and k. Consider

$$\vec{v}_j = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_{j-1}\vec{v}_{j-1}.$$

If we take the dot product of each side of this equation with $\vec{v}_j$ then, since the set of vectors is orthogonal, we get $\vec{v}_j \cdot \vec{v}_j = 0$, which contradicts the hypothesis that $\vec{v}_j \neq \vec{0}$. Therefore no $\vec{v}_j$ is a linear combination of its predecessors and by Page 203 Number 37, the set is independent. Therefore the set is a basis for its span. □

Theorem 6.3

Theorem 6.3. Projection Using an Orthogonal Basis.
Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$, and let $\vec{b} \in \mathbb{R}^n$. The projection of $\vec{b}$ on W is

$$\vec{b}_W = \text{proj}_W(\vec{b}) = \frac{\vec{b}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{b}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \cdots + \frac{\vec{b}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\vec{v}_k.$$

Proof. We know from Theorem 6.1 that $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$ where $\vec{b}_W$ is the projection of $\vec{b}$ on W and $\vec{b}_{W^\perp}$ is the projection of $\vec{b}$ on $W^\perp$. Since $\vec{b}_W \in W$ and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis of W, then

$$\vec{b}_W = r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k$$

for some scalars $r_1, r_2, \ldots, r_k$. We now find these r_i's.

Theorem 6.3 (continued)

Proof (continued). Taking the dot product of $\vec{b}$ with $\vec{v}_i$ we have

$$\begin{aligned}\vec{b}\cdot\vec{v}_i &= (\vec{b}_W + \vec{b}_{W^\perp})\cdot\vec{v}_i = (\vec{b}_W\cdot\vec{v}_i) + (\vec{b}_{W^\perp}\cdot\vec{v}_i)\\ &= (r_1\vec{v}_1\cdot\vec{v}_i + r_2\vec{v}_2\cdot\vec{v}_i + \cdots + r_k\vec{v}_k\cdot\vec{v}_i) + 0\\ &= r_i\vec{v}_i\cdot\vec{v}_i.\end{aligned}$$

Therefore $r_i = (\vec{b}\cdot\vec{v}_i)/(\vec{v}_i\cdot\vec{v}_i)$ and so

$$r_i\vec{v}_i = \frac{\vec{b}\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i}\vec{v}_i.$$

Substituting these values of the r_i's into the expression for $\vec{b}_W$ yields the theorem. □

Page 347 Number 4

Page 347 Number 4. Consider $W = \text{sp}([1,-1,1,1],[-1,1,1,1],[1,1,-1,1])$. Verify that the generating set of W is orthogonal and find the projection of $\vec{b} = [1,4,1,2]$ on W.

Solution. We check pairwise for orthogonality of the three generating vectors:

$$\begin{aligned}[1,-1,1,1]\cdot[-1,1,1,1] &= (1)(-1)+(-1)(1)+(1)(1)+(1)(1)\\ &= -1-1+1+1 = 0,\\ [1,-1,1,1]\cdot[1,1,-1,1] &= (1)(1)+(-1)(1)+(1)(-1)+(1)(1)\\ &= 1-1-1+1 = 0,\\ [-1,1,1,1]\cdot[1,1,-1,1] &= (-1)(1)+(1)(1)+(1)(-1)+(1)(1)\\ &= -1+1-1+1 = 0.\end{aligned}$$

Since each dot product is 0 then the vectors form an orthogonal set (in fact, an orthogonal basis for W, by Theorem 6.2, "Orthogonal Bases").

Page 347 Number 4 (continued)

Solution (continued). By Theorem 6.3, "Projection Using an Orthogonal Basis," we have the projection of $\vec{b}$ on W is

$$\vec{b}_W = \text{proj}_W(\vec{b}) = \frac{\vec{b}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{b}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \frac{\vec{b}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3}\vec{v}_3$$

where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are the three orthogonal generating vectors, so

$$\begin{aligned}\vec{b}_W &= \frac{[1,4,1,2]\cdot[1,-1,1,1]}{[1,-1,1,1]\cdot[1,-1,1,1]}[1,-1,1,1]\\ &\quad+\frac{[1,4,1,2]\cdot[-1,1,1,1]}{[-1,1,1,1]\cdot[-1,1,1,1]}[-1,1,1,1]\\ &\quad+\frac{[1,4,1,2]\cdot[1,1,-1,1]}{[1,1,-1,1]\cdot[1,1,-1,1]}[1,1,-1,1]\\ &= \frac{0}{4}[1,-1,1,1]+\frac{6}{4}[-1,1,1,]+\frac{6}{4}[1,1,-1,1]\end{aligned}$$

$$= 0[1,-1,1,1]+(3/2)[-1,1,1,1]+(3/2)[1,1,-1,1] = \boxed{[0,3,0,3].} \quad \square$$

Theorem 6.4

Theorem 6.4. Orthonormal Basis (Gram-Schmidt) Theorem. Let W be a subspace of $\mathbb{R}^n$, let $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_k\}$ be any basis for W, and let

$$W_j = \text{sp}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_j) \text{ for } j = 1,2,\dots,k.$$

Then there is an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \dots, \vec{q}_k\}$ for W such that $W_j = \text{sp}(\vec{q}_1, \vec{q}_2, \dots, \vec{q}_j)$.

Proof. We give a recursive construction which will reveal how to apply the Gram-Schmidt Process.

First, let $\vec{v}_1 = \vec{a}_1$ (we will create an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ and then normalize each $\vec{v}_i$ to create an orthonormal set). For $j = 2,3,\dots,k$, let $\vec{p}_j$ be the projection $\vec{a}_j$ on $W_{j-1} = \text{sp}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_{j-1})$ and let $\vec{v}_j = \vec{a}_j - \vec{p}_j$. This computation of $\vec{v}_j$ is given symbolically in Figure 6.7.

Theorem 6.4 (continued 1)

Proof (continued).

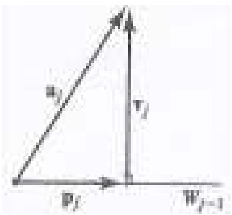


Figure 6.7

Since $\vec{p}_j$ is the projection of $\vec{a}_j$ on W_{j-1} then by Theorem 6.1(4), "Properties of $W^\perp$," and Definition 6.2, " Projection of $\vec{b}$ on W," we have

$$\vec{a}_j = (\vec{a}_j)_{W_{j-1}} + (\vec{a}_j)_{W_{j-1}^\perp} = \vec{p}_j + (\vec{a}_j - \vec{p}_j) = \vec{p}_j + \vec{v}_j$$

(and by Theorem 6.1(4), the choice of $\vec{p}_j$ and $\vec{v}_j$ are unique). Since $\vec{v}_j \in W_{j-1}^\perp$ then $\vec{v}_j$ is perpendicular to each $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{j-1} \in W_{j-1}$.

Theorem 6.4 (continued 2)

Theorem 6.4. Orthonormal Basis (Gram-Schmidt) Theorem. Let W be a subspace of $\mathbb{R}^n$, let $\{\vec{a_1}, \vec{a_2}, \dots, \vec{a_k}\}$ be any basis for W, and let

$$W_j = \text{sp}(\vec{a_1}, \vec{a_2}, \dots, \vec{a_j}) \text{ for } j = 1, 2, \dots, k.$$

Then there is an orthonormal basis $\{\vec{q_1}, \vec{q_2}, \dots, \vec{q_k}\}$ for W such that $W_j = \text{sp}(\vec{q_1}, \vec{q_2}, \dots, \vec{q_j})$.

Proof (continued). So each set $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_j}\}$ is an orthogonal set of vectors for each $j = 1, 2, \dots, k$ and since $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_j}\} \subset W_j$ (where $\dim(W_j) = j$) then by Theorem 6.2, "Orthogonal Bases," $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_j}\}$ is a basis for W_j.

Finally, define $\vec{q_i} = \vec{v_i}/\|\vec{v_i}\|$ for $i = 1, 2, \dots, j$. Then $W_j = \text{sp}(\vec{q_1}, \vec{q_2}, \dots, \vec{q_j})$, $\{\vec{q_1}, \vec{q_2}, \dots, \vec{q_j}\}$ is an orthonormal basis for W_j, and in particular $\{\vec{q_1}, \vec{q_2}, \dots, \vec{q_k}\}$ is an orthonormal basis for W, as claimed. □

Page 348 Number 10

Page 348 Number 10. Transform the basis $\{[1,1,1],[1,0,1],[0,1,1]\}$ for $\mathbb{R}^3$ into an orthonormal basis using the Gram-Schmidt Process.

Solution. First, denote the given basis vectors as $\vec{a}_1, \vec{a}_2, \vec{a}_3$ in order. Let $\vec{v_1} = \vec{a_1} = [1,1,1]$. Next, by the recursive formula above,

$$\vec{v_2} = \vec{a_2} - \frac{\vec{a_2}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} = [1,0,1] - \frac{[1,0,1]\cdot[1,1,1]}{[1,1,1]\cdot[1,1,1]}[1,1,1] = [1,0,1] - \frac{2}{3}[1,1,1]$$

$$= \left[\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right] = \frac{1}{3}[1,-2,1]$$

and

$$\vec{v_3} = \vec{a_3} - \frac{\vec{a_3}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} - \frac{\vec{a_3}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2}$$

$$= [0,1,1] - \frac{[0,1,1]\cdot[1,1,1]}{[1,1,1]\cdot[1,1,1]}[1,1,1] - \frac{[0,1,1]\cdot\frac{1}{3}[1,-2,2]}{\frac{1}{3}[1,-2,1]\cdot\frac{1}{3}[1,-2,1]}\frac{1}{3}[1,-2,1]$$

...

Page 348 Number 10 (continued 1)

Solution (continued). ...

$$= [0,1,1] - \frac{2}{3}[1,1,1] - \frac{-1}{6}[1,-2,1] = \left[-\frac{2}{3} + \frac{1}{6}, 1 - \frac{2}{3} - \frac{1}{3}, 1 - \frac{2}{3} + \frac{1}{6}\right]$$

$$= \left[-\frac{1}{2}, 0\frac{1}{2}\right] = \frac{1}{2}[-1,0,1].$$

Finally we normalize $\vec{v_1}, \vec{v_2}, \vec{v_3}$ to get

$$\vec{q_1} = \vec{v_1}/\|\vec{v_1}\| = \frac{[1,1,1]}{\|[1,1,1]\|} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right],$$

$$\vec{q_2} = \vec{v_2}/\|\vec{v_2}\| = \frac{\frac{1}{3}[1,-2,1]}{\|\frac{1}{3}[1,-2,1]\|} = \left[\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right],$$

$$\vec{q_3} = \vec{v_3}/\|\vec{v_3}\| = \frac{\frac{1}{2}[-1,0,1]}{\|\frac{1}{2}[-1,0,1]\|} = \left[\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right].$$

Page 348 Number 10 (continued 2)

Page 348 Number 10. Transform the basis $\{[1,1,1],[1,0,1],[0,1,1]\}$ for $\mathbb{R}^3$ into an orthonormal basis using the Gram-Schmidt Process.

Solution (continued). So an orthonormal basis is

$$\{\vec{q_1}, \vec{q_2}, \vec{q_3}\} = \left\{\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right], \left[\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right], \left[\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]\right\}.$$

□

Corollary 1

Corollary 1. ***QR*-Factorization.**
Let A be an $n \times k$ matrix with independent column vectors in $\mathbb{R}^n$. There exists an $n \times k$ matrix Q with orthonormal column vectors and an upper-triangular invertible $k \times k$ matrix R such that $A = QR$.

Proof. Denote the columns of A as $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$. In the proof of Theorem 6.4 we saw that there exists $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j\}$ and $\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_j\}$ both bases of $W_j = \text{sp}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_j)$. So each $\vec{a}_j$ is a unique linear combination of $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_j$:

$$\vec{a}_j = r_{1j}\vec{q}_1 + r_{2j}\vec{q}_2 + \cdots + r_{jj}\vec{q}_j \text{ for } j = 1, 2, \ldots, k.$$

Define $n \times k$ matrix Q with columns $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k$ and define $k \times k$ matrix $R = [r_{ij}]$ where the r_{ij} are the coefficients given above.

Corollary 1 (continued 1)

Proof (continued). Notice that

$$\begin{aligned}
\vec{a}_1 &= r_{11}\vec{q}_1 \\
\vec{a}_2 &= r_{12}\vec{q}_1 + r_{22}\vec{q}_2 \\
\vec{a}_3 &= r_{13}\vec{q}_1 + r_{23}\vec{q}_2 + r_{33}\vec{q}_3 \\
&\vdots \\
\vec{a}_k &= r_{1k}\vec{q}_1 + r_{2k}\vec{q}_2 + r_{3k}\vec{q}_3 + \cdots + r_{kk}\vec{q}_k
\end{aligned}$$

so that $r_{ij} = 0$ for $i > j$ and R is upper triangular:

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{kk} \end{bmatrix}.$$

Corollary 1 (continued 2)

Corollary 1. ***QR*-Factorization.**
Let A be an $n \times k$ matrix with independent column vectors in $\mathbb{R}^n$. There exists an $n \times k$ matrix Q with orthonormal column vectors and an upper-triangular invertible $k \times k$ matrix R such that $A = QR$.

Proof (continued). Since the columns of A are independent then $r_{ii} \neq 0$ for $i = 1, 2, \ldots, k$, and hence $\det(R) \neq 0$ and R^{-1} exists. Now if we let the ith column of R be vector $\vec{r}_i$ then $Q\vec{r}_i$ is a linear combination of $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k$ with coefficients $r_{1i}, r_{2i}, \ldots, r_{ki}$ (see Note 1.3.A) as

$$Q\vec{r}_i = r_{1i}\vec{q}_1 + r_{2i}\vec{q}_2 + \cdots + r_{ki}\vec{q}_{ki} = \vec{a}_i \text{ for } i = 1, 2, \ldots, k.$$

That is, the ith column of QR is $\vec{a}_i$ and this holds for $i = 1, 2, \ldots, k$. So $A = QR$, as claimed. □

Page 348 Number 26

Page 348 Number 26. Find a QR-factorization of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution. As seen in the proof of Corollary 1, we need to convert the columns of A, $\vec{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ into an orthonormal basis $\{\vec{q}_1, \vec{q}_2\}$ for $\text{sp}(\vec{a}_1, \vec{a}_2)$. We take $\vec{v}_1 = \vec{a}_1 = [0, 1, 0]^T$ and

$$\vec{v}_2 = \vec{a}_2 - \frac{\vec{a}_1 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{[1,1,1]^T \cdot [0,1,0]^T}{[0,1,0]^T \cdot [0,1,0]^T}[0,1,0]^T$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Page 348 Number 26 (continued)

Solution (continued). Then we take $\vec{q}_1 = \vec{v}_1/\|\vec{v}_1\| = [0,1,0]^T$ and $\vec{q}_2 = \vec{v}_2/\|\vec{v}_2\| = \frac{1}{\sqrt{2}}[1,0,1]^T$. So $Q = [\vec{q}_1\ \vec{q}_2] = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$. Next we need $\vec{a}_1$ and $\vec{a}_2$ as linear combinations of $\vec{q}_1$ and $\vec{q}_2$:

$$\vec{a}_1 = 1\vec{q}_1 + 0\vec{q}_2 \text{ (since } \vec{a}_1 = \vec{q}_1\text{); so } r_{11} = 1 \text{ and } r_{21} = 0.$$

Next, $\vec{a}_2 = r_{12}\vec{q}_1 + r_{22}\vec{q}_2$ or $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = r_{12}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r_{22}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, so clearly $r_{12} = 1$ and $r_{22} = \sqrt{2}$. Therefore $R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$.

So $A = QR$ where $\boxed{R = \begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}.}$ □

Corollary 2

Corollary 2. Expansion of an Orthogonal Set to an Orthogonal Basis. Every orthogonal set of vectors in a subspace W of $\mathbb{R}^n$ can be expanded if necessary to an orthogonal basis of W.

Proof. An orthogonal set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ of vectors in W is an independent set by Theorem 6.2, and can be expanded to a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r, \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_s\}$ of W by Theorem 2.3. We apply the Gram-Schmidt Process (Theorem 6.4) to this basis for W. Because the $\vec{v}_j$ are already mutually perpendicular, none of them will be changed by the Gram-Schmidt Process (since they are taken first), and so the process yields an orthogonal basis containing the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$. □

Page 348 Number 20

Page 348 Number 20. Find an orthonormal basis for $\mathbb{R}^3$ that contains the vector $(1/\sqrt{3})[1,1,1]$.

Solution. First we need a basis for $\mathbb{R}^3$ which includes $\frac{1}{\sqrt{3}}[1,1,1]$. So we consider the set $\left\{\frac{1}{\sqrt{3}}[1,1,1], [1,0,0], [0,1,0], [0,0,1]\right\}$. Of course, this set of 4 vectors from $\mathbb{R}^3$ must be dependent by Theorem 2.2, "Relative Sizes of Spanning and Independent Sets" (since $\mathbb{R}^3$ is dimension 3). We apply Theorem 2.1.A to find a basis for the span of the 4 vectors and row reduce a matrix with these vectors as columns:

$$\begin{bmatrix} 1/\sqrt{3} & 1 & 0 & 0 \\ 1/\sqrt{3} & 0 & 1 & 0 \\ 1/\sqrt{3} & 0 & 0 & 1 \end{bmatrix} \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \begin{bmatrix} 1/\sqrt{3} & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\underset{R_3 \to R_3 - R_12}{\overset{R_1 \to R_1 + R_2}{\sim}} \begin{bmatrix} 1/\sqrt{3} & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = H.$$

Page 348 Number 20 (continued 1)

Solution (continued). Since H is in row-echelon form and contains pivots in the first 3 columns then a basis for $\mathbb{R}^3$ is given by $\{(1/\sqrt{3})[1,1,1], [1,0,0], [0,1,0]\} = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. We now apply the Gram-Schmidt Process.

Let $\vec{v}_1 = \vec{a}_1 = (1/\sqrt{3})[1,1,1]$. Let

$$\begin{aligned} \vec{v}_2 &= \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \\ &= [1,0,0] - \frac{[1,0,0] \cdot \frac{1}{\sqrt{3}}[1,1,1]}{\frac{1}{\sqrt{3}}[1,1,1] \cdot \frac{1}{\sqrt{3}}[1,1,1]} \frac{1}{\sqrt{3}}[1,1,1] \\ &= [1,0,0] - \left(\frac{1}{3}\right)\left(\frac{1}{1}\right)[1,1,1] \\ &= \left[\frac{2}{3}, \frac{-1}{3}, \frac{-1}{3}\right] = \frac{1}{3}[2,-1,-1], \end{aligned}$$

Page 348 Number 20 (continued 2)

Solution (continued).

$$\begin{aligned}
\vec{v}_3 &= \vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 \\
&= [0,1,0] - \frac{[0,1,0]\cdot \frac{1}{\sqrt{3}}[1,1,1]}{\frac{1}{\sqrt{3}}[1,1,1]\cdot\frac{1}{\sqrt{3}}[1,1,1]}\frac{1}{\sqrt{3}}[1,1,1] \\
&\quad - \frac{[0,1,0]\cdot\frac{1}{3}[2,-1,-1]}{\frac{1}{3}[2,-1,-1]\cdot\frac{1}{2}[2,-1,-1]}\frac{1}{3}[2,-1,-1] \\
&= [0,1,0] - \left(\frac{1}{3}\right)\left(\frac{1}{1}\right)[1,1,1] - \left(\frac{1}{9}\right)\left(\frac{-1}{6/9}\right)[2,-1,-1] \\
&= \left[0 - \frac{1}{3} + \frac{2}{6}, 1 - \frac{1}{3} - \frac{1}{6}, 0 - \frac{1}{3} - \frac{1}{6}\right] = \left[0, \frac{1}{2}, \frac{-1}{2}\right] = \frac{1}{2}[0,1,-1].
\end{aligned}$$

So an orthogonal basis for $\mathbb{R}^3$ is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Page 348 Number 20 (continued 3)

Solution (continued). We normalize these vectors to get an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ (notice that $\|\vec{v}_1\| = 1$, so we take $\vec{q}_1 = \vec{v}_1$). So

$$\vec{q}_2 = \vec{v}_2/\|\vec{v}_2\| = \frac{\frac{1}{2}[2,-1,-1]}{\frac{1}{3}\sqrt{6}} = \frac{1}{\sqrt{6}}[2,-1,-1],$$

and

$$\vec{q}_3 = \vec{v}_3/\|\vec{v}_3\| = \frac{\frac{1}{2}[0,1,-1]}{\frac{1}{2}\sqrt{2}} = \frac{1}{\sqrt{2}}[0,1,-1].$$

So an orthonormal basis of $\mathbb{R}^3$ including $\vec{a}_1 = \vec{v}_1 = \vec{q}_1 = \frac{1}{\sqrt{3}}[1,1,1]$ is

$$\boxed{\left\{\frac{1}{\sqrt{3}}[1,1,1], \frac{1}{\sqrt{6}}[2,-1,-1], \frac{1}{\sqrt{2}}[0,1,-1]\right\}.} \ldots$$

Page 348 Number 20 (continued 4)

Page 348 Number 20. Find an orthonormal basis for $\mathbb{R}^3$ that contains the vector $(1/\sqrt{3})[1,1,1]$.

Solution (continued). Notice that this answer depends on the fact that we chose as a spanning set of $\mathbb{R}^3$ the given vector along with the standard basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$ of $\mathbb{R}^3$ (in this order). We could have chosen a different basis or the standard basis but in a different order and we would have gotten a different answer. There are an infinite number of correct answers. □

Page 349 Number 30

Page 349 Number 30. Let A be an $n \times n$ matrix. Prove that A has an orthonormal column vector if and only if A is invertible with inverse $A^{-1} = A^T$. HINT: Use Exercise 6.3.29 which states: "Let A be an $n \times k$ matrix. Prove that the column vectors of A are orthonormal if and only if $A^T A = \mathcal{I}$." NOTE: Exercise 6.3.29 is the inspiration for the definition of "orthogonal matrix" in the next section.

Solution. By Exercise 6.3.29 (with $k = n$) we have that the column vectors of A are orthonormal if and only if $A^T A = \mathcal{I}$. Notice that, since A and A^T are $n \times n$ matrices, by Theorem 1.11, "A Commutivity Property," $A^T A = \mathcal{I}$ implies $AA^T = \mathcal{I}$. So if the column vectors of A are orthonormal then, by Exercise 6.3.29, $A^T A = \mathcal{I} = AA^T$ and so A is invertible with $A^{-1} = A^T$. Conversely, suppose A is invertible and $A^{-1} = A^T$. Then $A^{-1}A = A^T A = \mathcal{I}$ and so by Exercise 6.3.29 the column vectors of A are orthonormal. □

Page 349 Number 32

Page 349 Number 32. Let V be an inner-product space of dimension n and let B be an ordered orthonormal basis for V. Prove that, for any vectors $\vec{a}, \vec{c} \in V$, the inner product $\langle \vec{a}, \vec{c} \rangle$ is equal to dot product of the coordinate vectors of $\vec{a}$ and $\vec{c}$ relative to B. NOTE: We already know that any two n-dimensional vector spaces are isomorphic by the "Fundamental Theorem of Finite Dimensional Vector Spaces," Theorem 3.3.A, and the isomorphism involves mapping each vector of a given n-dimensional vector space to its coordinate vector in $\mathbb{R}^n$. This exercise shows that the inner product structures is also preserved under the same isomorphism so that we can conclude that any two n-dimensional inner product spaces are isomorphic (and so any n-dimensional inner product space is isomorphic to $\mathbb{R}$ where the inner product on $\mathbb{R}^n$ is the usual dot product).

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Proof. Let ordered basis $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$, $\vec{a} = a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n$, and $\vec{c} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$, so that the coordinate vectors are $\vec{a}_B = [a_1, a_2, \ldots, a_n]$ and $\vec{c}_B = [c_1, c_2, \ldots, c_n]$. We apply the properties of an inner product given in Definition 3.1.2 to get

$$\begin{aligned}
\langle \vec{a}, \vec{c} \rangle &= \langle a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n, c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \rangle \\
&= \langle a_1\vec{b}_1, c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \rangle \\
&\quad + \langle a_2\vec{b}_2, c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \rangle + \cdots \\
&\quad + \langle a_n\vec{b}_n, c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \rangle \\
&= \langle a_1\vec{b}_1 \rangle + \langle a_1\vec{b}_1, c_2\vec{b}_2 \rangle + \cdots + \langle a_1\vec{b}_1, c_n\vec{v}_n \rangle \\
&\quad + \langle a_2\vec{b}_2, c_1\vec{b}_1 \rangle + \langle a_2\vec{b}_2, c_2\vec{b}_2 \rangle + \cdots + \langle a_2\vec{b}_2, c_n\vec{b}_n \rangle + \cdots \\
&\quad + \langle a_n\vec{b}_n, c_1\vec{b}_1 \rangle + \langle a_n\vec{b}_n, c_2\vec{b}_2 \rangle + \cdots + \langle a_n\vec{b}_n, c_n\vec{b}_n \rangle
\end{aligned}$$

Page 349 Number 32 (continued 2)

Proof (continued).

$$\begin{aligned}
\langle \vec{a}, \vec{c} \rangle &= a_1c_1\langle \vec{b}_1, \vec{b}_1 \rangle + a_1c_2\langle \vec{b}_1, \vec{b}_2 \rangle + \cdots a_1c_n\langle \vec{b}_1, \vec{b}_n \rangle \\
&\quad + a_2c_1\langle \vec{b}_2, \vec{b}_1 \rangle + a_2c_2\langle \vec{b}_2, \vec{b}_2 \rangle + \cdots a_2c_n\langle \vec{b}_2, \vec{b}_n \rangle + \cdots \\
&\quad + a_nc_1\langle \vec{b}_n, \vec{b}_1 \rangle + a_nc_2\langle \vec{b}_n, \vec{b}_2 \rangle + \cdots a_nc_n\langle \vec{b}_n, \vec{b}_n \rangle \\
&= a_1c_1 + 0 + 0 + \cdots + 0 \\
&\quad + 0 + a_2c_2 + \cdots + 0 \\
&\quad + 0 + 0 + \cdots + a_nc_n \\
&= a_1c_1 + a_2c_2 + \cdots + a_nc_n = \vec{a}_B \cdot \vec{c}_B.
\end{aligned}$$

□

Page 349 Number 34

Page 349 Number 34. Find an orthonormal basis for $\text{sp}(1, x, x^2)$ for $-1 \le x \le 1$ if the inner product is defined by $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

Solution. We apply the Gram-Schmidt Process to $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \{1, x, x^2\}$. We simply replace the dot product of $\mathbb{R}^n$ with the inner product given here. Let $\vec{v}_1 = \vec{a}_1 = 1$. Then

$$\begin{aligned}
\vec{v}_2 &= \vec{a}_2 - \frac{\langle \vec{a}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 = x - \left(\frac{\int_{-1}^{1} x \cdot 1\,dx}{\int_{-1}^{1} 1 \cdot 1\,dx} \right) 1 \\
&= x - \left(\frac{\frac{1}{2}x^2|_{-1}^{1}}{x|_{-1}^{1}} \right)(1) = x - \frac{\frac{1}{2}(1)^2 - \frac{1}{2}(-1)^2}{(1) - (-1)} = x - 0 = x,
\end{aligned}$$

and

$$\vec{v}_3 = \vec{a}_3 - \frac{\langle \vec{a}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 - \frac{\langle \vec{a}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}\vec{v}_2$$

...

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Solution (continued). ...

$$\begin{aligned}\vec{v}_3 &= x^2 - \left(\frac{\int_{-1}^{1} x^2 \cdot 1\, dx}{\int_{-1}^{1} 1 \cdot 1\, dx}\right) 1 - \left(\frac{\int_{-1}^{1} x^2 \cdot x\, dx}{\int_{-1}^{1} x \cdot x\, dx}\right) x \\ &= x^2 - \left(\frac{\frac{1}{3}x^3|_{-1}^{1}}{x|_{-1}^{1}}\right) 1 - \left(\frac{\frac{1}{4}x^4|_{-1}^{1}}{\frac{1}{3}(1)^3 - \frac{1}{3}(-1)^3}\right) x \\ &= x^2 - \left(\frac{1}{3}\right) 1 - (0)x = x^2 - \frac{1}{3}.\end{aligned}$$

Finally, we normalize:

$$\vec{q}_1 = \vec{v}_1/\|\vec{v}_1\| = \frac{1}{\sqrt{\langle 1, 1\rangle}} = \frac{1}{\sqrt{\int_{-1}^{1} 1\, dx}} = \frac{1}{\sqrt{x|_{-1}^{1}}} = \frac{1}{\sqrt{(1) - (-1)}} = \frac{1}{\sqrt{2}},$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{x}{\sqrt{\langle x, x\rangle}} = \frac{x}{\sqrt{\int_{-1}^{1} x^2\, dx}} = \frac{x}{\sqrt{\frac{1}{3}x^3|_{-1}^{1}}} = \dots$$

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Solution (continued). ...

$$\vec{q}_2 = \frac{x}{\sqrt{\frac{1}{3}(1)^3 - \frac{1}{3}(-1)^3}} = \frac{x}{\sqrt{\frac{2}{3}}} = \frac{\sqrt{3}x}{\sqrt{2}}$$

and

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{x^2 - \frac{1}{3}}{\sqrt{\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle}} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} (x^2 - 1/3)^2\, dx}} \\ &= \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} (x^4 - \frac{2}{3}x^2 - \frac{1}{9})\, dx}} = \frac{x^2 - \frac{1}{3}}{\sqrt{\left(\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right)|_{-1}^{1}}} \\ &= \frac{x^2 - \frac{1}{3}}{\sqrt{\left(\frac{1}{5}(1)^5 - \frac{2}{9}(1)^3 + \frac{1}{9}(1)\right) - \left(\frac{1}{5}(-1)^5 - \frac{2}{9}(-1)^3 + \frac{1}{9}(-1)\right)}} \\ &= \frac{x^2 - \frac{1}{3}}{\sqrt{8/45}} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) = \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right).\end{aligned}$$

Page 349 Number 34 (continued 3)

Page 349 Number 34. Find an orthonormal basis for $\text{sp}(1, x, x^2)$ for $-1 \le x \le 1$ if the inner product is defined by $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx$.

Solution (continued). So an orthonormal basis for $\text{sp}(1, x, x^2)$ is

$$\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}x}{\sqrt{2}}, \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right)\right\}.$$

□