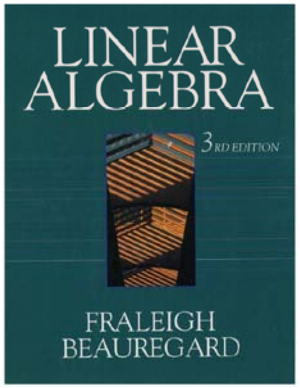
Linear Algebra

Chapter 6: Orthogonality

Section 6.3. Orthogonal Matrices—Proofs of Theorems



Theorem 6.5

Theorem 6.5. Characterizing Properties of an Orthogonal Matrix. Let A be an $n \times n$ matrix. The following conditions are equivalent:

1. The rows of A form an orthonormal basis for $\mathbb{R}^n$.
2. The columns of A form an orthonormal basis for $\mathbb{R}^n$.
3. The matrix A is orthogonal — that is, A is invertible and $A^{-1} = A^T$.

Proof. Suppose the columns of A are vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$. Then A is orthogonal if and only if

$$\mathcal{I} = A^T A = \begin{bmatrix} \cdots & \vec{a}_1 & \cdots \\ \cdots & \vec{a}_2 & \cdots \\ & \vdots & \\ \cdots & \vec{a}_n & \cdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

and we see that the diagonal entries of the product are $\vec{a}_j \cdot \vec{a}_j = 1$ therefore each vector is a unit vector.

Theorem 6.5 (continued)

Theorem 6.5. Characterizing Properties of an Orthogonal Matrix. Let A be an $n \times n$ matrix. The following conditions are equivalent:

1. The rows of A form an orthonormal basis for $\mathbb{R}^n$.
2. The columns of A form an orthonormal basis for $\mathbb{R}^n$.
3. The matrix A is orthogonal — that is, A is invertible and $A^{-1} = A^T$.

Proof (continued). All off-diagonal entries of $\mathcal{I}$ are 0 and so for $i \neq j$ we have $\vec{a}_i \cdot \vec{a}_j = 0$. Therefore the columns of A are orthonormal (and conversely if the columns of A are orthonormal then $A^T A = \mathcal{I}$). Now $A^T = A^{-1}$ if and only if A is orthogonal, so A is orthogonal if and only if $AA^T = \mathcal{I}$ or $(A^T)^T A^T = \mathcal{I}$. So A is orthogonal if and only if A^T is orthogonal, and hence the rows of A are orthonormal if and only if A is orthogonal. □

Page 358 Number 2

Page 358 Number 2. Verify that the matrix $A = \begin{bmatrix} 3/5 & 0 & 4/5 \\ -4/5 & 0 & 3/5 \\ 0 & 1 & 0 \end{bmatrix}$ is orthogonal and find A^{-1}.

Solution. By Definition 6.4, "Orthogonal Matrix," we need to check if $A^T A = \mathcal{I}$. So

$$A^T A = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 0 & 0 & 1 \\ 4/5 & 3/5 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & 0 & 4/5 \\ -4/5 & 0 & 3/5 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (3/5)^2 + (-4/5)^2 + 0 & 0+0+0 & (3/5)(4/5) + (-4/5)(3/5) + 0 \\ 0+0+0 & 0+0+1 & 0+0+0 \\ (4/5)(3/5) + (3/5)(-4/5) + 0 & 0+0+0 & (4/5)^2 + (3/5)^2 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathcal{I}.$$

Page 358 Number 2 (continued)

Page 358 Number 2. Verify that the matrix $A = \begin{bmatrix} 3/5 & 0 & 4/5 \\ -4/5 & 0 & 3/5 \\ 0 & 1 & 0 \end{bmatrix}$ is orthogonal and find A^{-1}.

Solution (continued). So A is orthogonal and

$$\boxed{A^{-1} = A^T = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 0 & 0 & 1 \\ 4/5 & 3/5 & 0 \end{bmatrix}}. \quad \square$$

Theorem 6.6

Theorem 6.6. Properties of $A\vec{x}$ for an Orthogonal Matrix A.
Let A be an orthogonal $n \times n$ matrix and let $\vec{x}$ and $\vec{y}$ be any column vectors in $\mathbb{R}^n$. Then

1. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$,
2. $\|A\vec{x}\| = \|\vec{x}\|$, and
3. The angle between nonzero vectors $\vec{x}$ and $\vec{y}$ equals the angle between $A\vec{x}$ and $A\vec{y}$.

Proof. Recall that $\vec{x} \cdot \vec{y} = (\vec{x}^T)\vec{y}$. Then since A is orthogonal,

$$[(A\vec{x}) \cdot (A\vec{y})] = (A\vec{x})^T A\vec{y} = \vec{x}^T A^T A\vec{y} = \vec{x}^T \mathcal{I}\vec{y} = \vec{x}^T\vec{y} = [\vec{x} \cdot \vec{y}]$$

and the first property is established.

For the second property,

$$\|A\vec{x}\| = \sqrt{A\vec{x} \cdot A\vec{x}} = \sqrt{\vec{x} \cdot \vec{x}} = \|\vec{x}\|.$$

Theorem 6.6 (continued)

Theorem 6.6. Properties of $A\vec{x}$ for an Orthogonal Matrix A.
Let A be an orthogonal $n \times n$ matrix and let $\vec{x}$ and $\vec{y}$ be any column vectors in $\mathbb{R}^n$. Then

1. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$,
2. $\|A\vec{x}\| = \|\vec{x}\|$, and
3. The angle between nonzero vectors $\vec{x}$ and $\vec{y}$ equals the angle between $A\vec{x}$ and $A\vec{y}$.

Proof (continued). Since dot products and norms are preserved under multiplication by A, then the angle

$$\cos^{-1}\left(\frac{\vec{x} \cdot \vec{y}}{\sqrt{\vec{x} \cdot \vec{x}}\sqrt{\vec{y} \cdot \vec{y}}}\right) = \cos^{-1}\left(\frac{(A\vec{x}) \cdot (A\vec{y})}{\sqrt{(A\vec{x}) \cdot (A\vec{x})}\sqrt{(A\vec{y}) \cdot (A\vec{y})}}\right).$$

$\square$

Page 358 Number 12

Page 358 Number 12. Let $B = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ be an ordered orthonormal basis for $\mathbb{R}^4$, and let $\vec{b}_B = [2, 1, 4, -3]$ be the coordinate vector of a vector $\vec{b}$ in $\mathbb{R}^4$ relative to B. Find $\|\vec{b}\|$ and explain.

Solution. Define 4×4 matrix A with $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ as its columns, $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4]$. Then by Theorem 6.5, "Characterizing Properties of an Orthogonal Matrix" (the (2) implies (3) part), A is an orthogonal matrix. So, by Note 1.3.A, $A\vec{b}_B = 2\vec{a}_1 + \vec{a}_2 + 4\vec{a}_3 - 3\vec{a}_4 = \vec{b}$ and by Theorem 6.6(2), "Properties of $A\vec{x}$ for an Orthogonal Matrix A,"

$$\|A\vec{b}_B\| = \|\vec{b}_B\| = \|[2, 1, 4, -3]\| = \sqrt{(2)^2 + (1)^2 + (4)^2 + (-3)^2} = \sqrt{30}.$$

Since $A\vec{b}_B = \vec{b}$, then $\|\vec{b}\| = \|A\vec{b}_B\| = \boxed{\sqrt{30}.}$ $\square$

Theorem 6.7

Theorem 6.7. Orthogonality of Eigenspaces of a Real Symmetric Matrix.
Eigenvectors of a real symmetric matrix that correspond to different eigenvalues are orthogonal. That is, the eigenspaces of a real symmetric matrix are orthogonal.

Proof. Let A be an $n \times n$ symmetric matrix, and let $\vec{v_1}$ and $\vec{v_2}$ be eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2, respectively. Then

$$A\vec{v_1} = \lambda_1 \vec{v_1} \text{ and } A\vec{v_2} = \lambda_2 \vec{v_2}.$$

We need to show that $\vec{v_1}$ and $\vec{v_2}$ are orthogonal. Notice that

$$\lambda_1(\vec{v_1} \cdot \vec{v_2}) = (\lambda_1 \vec{v_1}) \cdot \vec{v_2} = (A\vec{v_1}) \cdot \vec{v_2} = (A\vec{v_1})^T \vec{v_2} = (\vec{v_1}^{\,T} A^T)\vec{v_2}.$$

Similarly

$$[\lambda_2(\vec{v_1} \cdot \vec{v_2})] = \vec{v_1}^{\,T} A \vec{v_2}.$$

Theorem 6.7 (continued)

Theorem 6.7. Orthogonality of Eigenspaces of a Real Symmetric Matrix.
Eigenvectors of a real symmetric matrix that correspond to different eigenvalues are orthogonal. That is, the eigenspaces of a real symmetric matrix are orthogonal.

Proof (continued). Since A is symmetric, then $A = A^T$ and

$$\lambda_1(\vec{v_1} \cdot \vec{v_2}) = \lambda_2(\vec{v_1} \cdot \vec{v_2}) \text{ or } (\lambda_1 - \lambda_2)(\vec{v_1} \cdot \vec{v_2}) = 0.$$

Since $\lambda_1 - \lambda_2 \neq 0$, then it must be the case that $\vec{v_1} \cdot \vec{v_2} = 0$ and hence $\vec{v_1}$ and $\vec{v_2}$ are orthogonal. □

Theorem 6.8

Theorem 6.8. Fundamental Theorem of Real Symmetric Matrices.
Every real symmetric matrix A is diagonalizable. The diagonalization $C^{-1}AC = D$ can be achieved by using a real orthogonal matrix C.

Proof. By Theorem 5.5, matrix A has only real roots of its characteristic polynomial and the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity. Therefore we can find a basis for $\mathbb{R}^n$ which consists of eigenvectors of A. Next, we can use the Gram-Schmidt Process to create an orthonormal basis for each eigenspace. We know by Theorem 6.7, "Orthogonality of Eigenspaces of a Real Symmetric Matrix," that the basis vectors from different eigenspaces are perpendicular, and so we have a basis of mutually orthogonal eigenvectors of unit length. By Theorem 5.2, "Matrix Summary of Eigenvalues of A," we make matrix C by using these unit eigenvectors as columns and we have that $C^{-1}AC = D$ where D consists of the eigenvalues of A. Since the columns of C form an orthonormal set, matrix C is a real orthogonal matrix, as claimed. □

Page 359 Number 16

Page 359 Number 16. Find a matrix C such that $D = C^{-1}AC$ is an orthogonal diagonlization of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Solution. Notice that A is symmetric, so by Theorem 6.8, "Fundamental Theorem of Real Symmetric Matrices," such a matrix C exists. By Theorem 5.2, "Matrix Summary of Eigenvalues of A," we need the eigenvalues of A since they will determine matrix D. Consider

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} 0-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{vmatrix}$$

$$= (-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + (1)\begin{vmatrix} 1 & 2-\lambda \\ 1 & 1 \end{vmatrix}$$

Page 359 Number 16 (continued 1)

Solution (continued.

$$= (-\lambda)\left((2-\lambda)(-\lambda)-(1)(1)\right)-(1)\left((1)(-\lambda)-(1)(1)\right)$$
$$+(1)\left((1)(1)-(2-\lambda)(1)\right)$$
$$= -\lambda(-2\lambda+\lambda^2-1)+(\lambda+1)+(-1+\lambda) = 2\lambda^2-\lambda^3+\lambda+\lambda+1-1+\lambda$$
$$= -\lambda^3+2\lambda^2+3\lambda = -\lambda(\lambda^2-2\lambda-3) = -\lambda(\lambda-3)(\lambda+1).$$

Therefore the eigenvalues of A are $\lambda_1=-1$, $\lambda_2=0$, and $\lambda_3=3$. Since A is real and symmetric and the eigenvalues of A are different, then by Theorem 6.7, "Orthogonality of Eigenspaces of a Real Symmetric Matrix," if we find an eigenvalue for each of the three eigenvalues then the three eigenvectors will form an orthogonal set. We can then normalize these three eigenvectors to produce a set of three orthonormal eigenvectors (instead, we will carefully choose the eigenvectors to be unit vectors so the normalization will not be necessary). These eigenvectors then form the columns of matrix C by Theorem 5.2.

Page 359 Number 16 (continued 2)

Solution (continued). So for eigenvectors, consider:

$\underline{\lambda_1=-1.}$ Consider the system of equations $A\vec{x}=\lambda_1\vec{x}$ or $(A-\lambda_1\mathcal{I})\vec{x}=\vec{0}$:

$$[A-\lambda_1\mathcal{I}\mid\vec{0}] = \left[\begin{array}{ccc|c} 0-(-1) & 1 & 1 & 0 \\ 1 & 2-(-1) & 1 & 0 \\ 1 & 1 & 0-(-1) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\underset{R_3\to R_3-R_1}{\overset{R_2\to R_2-R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_2\to R_2/2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{R_1\to R_1-R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so we need $\begin{array}{rcl} x_1 + x_3 & = & 0 \\ x_2 & = & 0 \end{array}$ or $\begin{array}{rcl} x_1 & = & -x_3 \\ x_2 & = & 0 \\ x_3 & = & x_3 \end{array}$ or, ...

Page 359 Number 16 (continued 3)

Solution (continued). ...with $r=x_3$ as a free variable, $\begin{array}{rcl} x_1 & = & -r \\ x_2 & = & 0 \\ x_3 & = & r \end{array}$ or

$\vec{x} = r\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ where $r\in\mathbb{R}$. So we can choose an $r\neq 0$ to get an eigenvector corresponding to $\lambda_1=-1$. We take $r=1/\sqrt{2}$ to get eigenvector $\vec{x}_1=[-1/\sqrt{2},0,1/\sqrt{2}]^T$ (notice that by the choice of r, $\|\vec{x}_1\|=1$ and $\vec{x}_1$ is a unit vector).

$\underline{\lambda_2=0.}$ Consider the system of equations $A\vec{x}=\lambda_2\vec{x}$ or $(A-\lambda_2\mathcal{I})\vec{x}=\vec{0}$:

$$[A-\lambda_2\mathcal{I}\mid\vec{0}] = \left[\begin{array}{ccc|c} 0-(0) & 1 & 1 & 0 \\ 1 & 2-(0) & 1 & 0 \\ 1 & 1 & 0-(0) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

Page 359 Number 16 (continued 4)

Solution (continued).

$$\overset{R_1\leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \overset{R_2\to R_2-R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\underset{R_3\to R_3-R_2}{\overset{R_1\to R_1-R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so we need $\begin{array}{rcl} x_1 - x_3 & = & 0 \\ x_2 + x_3 & = & 0 \end{array}$ or $\begin{array}{rcl} x_1 & = & x_3 \\ x_2 & = & -x_3 \\ x_3 & = & x_3 \end{array}$ or, with $r=x_3$ as a free variable, $\begin{array}{rcl} x_1 & = & r \\ x_2 & = & -r \\ x_3 & = & r \end{array}$ or $\vec{x} = r\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ where $r\in\mathbb{R}$. So we can choose an $r\neq 0$ to get an eigenvector corresponding to $\lambda_2=0$.

Page 359 Number 16 (continued 5)

Solution (continued). We take $r = 1/\sqrt{3}$ to get eigenvector $\vec{x}_2 = [-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}]^T$ (notice that by the choice of r, $\|\vec{x}_2\| = 1$ and $\vec{x}_2$ is a unit vector).

$\underline{\lambda_3 = 3.}$ Consider the system of equations $A\vec{x} = \lambda_3\vec{x}$ or $(A - \lambda_3\mathcal{I})\vec{x} = \vec{0}$:

$$[A-\lambda_3\mathcal{I} \mid \vec{0}] = \left[\begin{array}{ccc|c} 0-(3) & 1 & 1 & 0 \\ 1 & 2-(3) & 1 & 0 \\ 1 & 1 & 0-(3) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -3 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right]$$

$$\underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right] \underset{R_3 \to R_3 - R_1}{\underset{\sim}{R_2 \to R_2 + 3R_1}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$$

$$\underset{\sim}{R_2 \to R_2/(-2)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \underset{R_3 \to R_3 - 2R_2}{\underset{\sim}{R_1 \to R_1 + R_2}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

Page 359 Number 16 (continued 6)

Solution (continued). . . . so we need $\begin{array}{rcrcl} x_1 & & - & x_3 & = 0 \\ & x_2 & - & 2x_3 & = 0 \end{array}$ or

$$\begin{array}{rcl} x_1 & = & x_3 \\ x_2 & = & 2x_3 \\ x_3 & = & x_3 \end{array} \text{ or, with } r = x_3 \text{ as a free variable, } \begin{array}{rcl} x_1 & = & r \\ x_2 & = & 2r \\ x_3 & = & r \end{array} \text{ or}$$

$\vec{x} = r\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ where $r \in \mathbb{R}$. So we can choose an $r \neq 0$ to get an eigenvector corresponding to $\lambda_3 = 3$. We take $r = 1/\sqrt{6}$ to get eigenvector $\vec{x}_3 = [1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6}]^T$ (notice that by the choice of r, $\|\vec{x}_3\| = 1$ and $\vec{x}_3$ is a unit vector).

Page 359 Number 16 (continued 7)

Solution (continued). So we take

$$C = [\vec{x}_1\ \vec{x}_2\ \vec{x}_3] = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

so that (since C is orthogonal)

$$C^{-1} = C^T = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}.$$

Then $D = C^{-1}AC = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. □

Theorem 6.9

Theorem 6.9. Orthogonal Transformations vis-à-vis Matrices. A linear transformation T of $\mathbb{R}^n$ into itself is orthogonal if and only if its standard matrix representation A is an orthogonal matrix.

Proof. By definition, T preserves dot products if and only if it is orthogonal, and so its standard matrix A must preserve dot products and so by Theorem 6.5 A is orthogonal. Conversely, we know that the columns of A are $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ where $\vec{e}_j$ is the jth unit coordinate vector of $\mathbb{R}^n$, by Theorem 3.10. We have

$$T(\vec{e}_i) \cdot T(\vec{e}_j) = \vec{e}_i \cdot \vec{e}_j = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$$

and so the columns of A form an orthonormal basis of $\mathbb{R}^n$. So A is an orthogonal matrix. □

Page 359 Number 34

Page 359 Number 34. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T([x, y]) = [2x, y]$. Determine whether T is orthogonal or not.

Solution. We apply Theorem 6.9, "Orthogonal Transformations vis-à-vis Matrices." Consider $T(\hat{\imath}) = T([1, 0]) = [2(1), 0] = [2, 0]$ and $T(\hat{\jmath}) = T([0, 1]) = [2(0), 1] = [0, 1]$. Then the standard matrix representation of T (by the Corollary in Section 2.3, "Linear Transformations of Euclidean Spaces") is $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Since the first column of A is not a unit vector and so A is not an orthogonal matrix. So, by Theorem 6.9, $\boxed{T \text{ is not an orthogonal linear transformation.}}$ □

Page 359 Number 20

Page 359 Number 20. Let A be an orthonormal $n \times n$ matrix. prove that $\|A\vec{x}\| = \|A^{-1}\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$.

Proof. Since A is orthogonal, then by Theorem 6.5, "Characterizing Properties of an Orthogonal Matrix" (the (3) implies (1) part), the rows of A form an orthonormal basis for $\mathbb{R}^n$. So the columns of A^T form an orthonormal basis for $\mathbb{R}^n$ and again by Theorem 6.5 (the (2) implies (3) part) A^T is orthogonal. Since A is orthogonal then by Definition 6.4, "Orthogonal Matrix," $A^{-1} = A^T$. So both A and A^T are orthogonal and by Theorem 6.6.2, "Properties of $A\vec{x}$ for an Orthogonal Matrix A" (Preservation fo length), $\|A\vec{x}\| = \|\vec{x}\|$ and $\|A^{-1}\vec{x}\| = \|\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$. Therefore $\|A\vec{x}\| = \|A^{-1}\vec{x}\|$, as claimed. □

Page 359 Number 24

Page 359 Number 24. Let $D = C^{-1}AC$ be a diagonal matrix wher C is an orthogonal matrix. Prove that A is symmetric.

Proof. Since C is orthogonal then by Definition 6.4, "Orthogonal Matrix," $C^{-1} = C^T$. Since D is diagonal then it is symmetric and so $D^T = D$. So $CDC^{-1} = A$ or $CDC^T = A$ and

$$\begin{aligned} A^T &= (CDC^T)^T \\ &= (C^T)^T D^T C^T \text{ by Notes 1.3.B} \\ &= CDC^{-1} = A \end{aligned}$$

and so A is symmetric (by Definition 1.11, "Symmetric Matrix"), as claimed. □

Page 359 Number 30

Page 359 Number 30. Let $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ be an orthonormal basis of column vectors for $\mathbb{R}^n$, and let C be an orthogonal $n \times n$ matrix. Prove that $\{C\vec{a}_1, C\vec{a}_2, \ldots, C\vec{a}_n\}$ is also an orthonormal basis for $\mathbb{R}^n$.

Proof. Since $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ is an orthonormal set then $\vec{a}_i \cdot \vec{a}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ (since $\vec{a}_i \cdot \vec{a}_j = \|\vec{a}_i\| = 1$ for $i = 1, 2, \ldots, n$). Let $C\vec{a}_i, C\vec{a}_j \in \{C\vec{a}_1, C\vec{a}_2, \ldots, C\vec{a}_n\}$. Then since C is orthogonal then

$$\begin{aligned} (C\vec{a}_i) \cdot (C\vec{a}_j) &= \vec{a}_i \cdot \vec{a}_j \text{ by Theorem 6.6(1), "Properties of } A\vec{x} \text{ for an} \\ &\quad \text{Orthogonal Matrix } A\text{" (Preservation of Dot Product)} \\ &= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}. \end{aligned}$$

Page 359 Number 30 (continued)

Page 359 Number 30. Let $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ be an orthonormal basis of column vectors for $\mathbb{R}^n$, and let C be an orthogonal $n \times n$ matrix. Prove that $\{C\vec{a}_1, C\vec{a}_2, \ldots, C\vec{a}_n\}$ is also an orthonormal basis for $\mathbb{R}^n$.

Proof (continued). So $\{C\vec{a}_1, C\vec{a}_2, \ldots, C\vec{a}_n\}$ is a basis by hypothesis then each $\vec{a}_i$ is nonzero. Since C is orthogonal then it is invertible (and $C^{-1} = C^T$ by Definition 6.4, "Orthogonal Matrix") and each $C\vec{a}_i$ is nonzero. So by Theorem 6.2, "Orthogonal Bases," $\{C\vec{a}_1, C\vec{a}_2, \ldots, C\vec{a}_n\}$ is a basis for $\mathbb{R}^n$, as claimed. □