Linear Algebra

Chapter 6: Orthogonality

Section 6.5. The Method of Least Squares—Proofs of Theorems

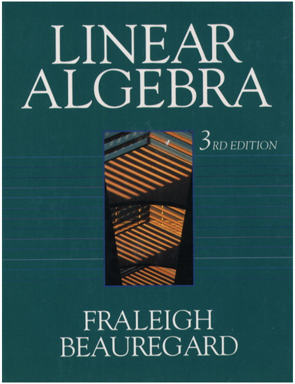

Table of contents

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Proof. By Theorem 6.1, $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$ where $\vec{b}_W \in W$ and $\vec{b}_{W^\perp} \in W^\perp$. To minimize $\|\vec{b} - \vec{w}\|$ we consider $\|\vec{b} - \vec{w}\|^2$:

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$$\begin{aligned}
\|\vec{b} - \vec{w}\|^2 &= (\vec{b} - \vec{w}) \cdot (\vec{b} - \vec{w}) \\
&= \Big((\vec{b} - \vec{b}_W) + (\vec{b}_W - \vec{w})\Big) \cdot \Big((\vec{b} - \vec{b}_W) + (\vec{b}_W - \vec{w})\Big) \\
&= (\vec{b} - \vec{b}_W) \cdot (\vec{b} - \vec{b}_W) + 2(\vec{b} - \vec{b}_W) \cdot (\vec{b}_W - \vec{w}) \\
&\quad + (\vec{b}_W - \vec{w}) \cdot (\vec{b}_W - \vec{w}) \\
&= \|\vec{b} - \vec{b}_W\|^2 + 2\vec{b}_{W^\perp} \cdot (\vec{b}_W - \vec{w}) + \|\vec{b}_W - \vec{w}\|^2 \\
&= \|\vec{b} - \vec{b}_W\|^2 + \|\vec{b}_W - \vec{w}\|^2 \text{ since } \vec{b}_{W^\perp} \perp (\vec{b}_W - \vec{w}) \\
&\quad \text{because } \vec{b}_W - \vec{w} \in W \ldots
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Proof(continued). ...

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So $\|\vec{b} - \vec{w}\| \geq \|\vec{b} - \vec{b}_W\|$ and $\vec{w} = \vec{b}_W$ minimizes the quantity $\|\vec{b} - \vec{w}\|$ where $\vec{w} \in W$. From the computation we see that the minimum is attained only when $vecw = \vec{b}_W$, so the choice of $\vec{w}$ is unique. □

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