1. Are these vectors in $\mathbb{R}^3$ dependent or independent: $\{(1, -4, 3), (3, -11, 2), (1, -3, -4)\}$? Explain.

HINT:

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 3 & -11 & 2 \\ 1 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -25 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{bmatrix} = H \text{ and } B = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -11 & -3 \\ 3 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = J.$$

We create a matrix with the given vectors as columns (so we consider matrix $B$), so that the span of the three vectors is the column space of $B$. We find a basis for the column space of $B$ by using the columns of $B$ for which the reduced row echelon form $J$ have corresponding columns with pivots. Since the first and second columns of $J$ contain pivots, then the first and second columns of $B$ form a basis for the column space. Hence a basis for the span of the three given vectors in $\{(1, -4, 3), (3, -11, 2)\}$.

Since a basis for the span does not include all three vectors, then the set of the given three vectors is NOT a basis (that is, NOT a linearly independent spanning set). Of course, it is a spanning set for the space it spans, so it must not be linearly independent.
2. Consider \( A = \begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix} \). Find (a) the rank of \( A \), (b) a basis for the row space of \( A \), and (c) a basis for column space of \( A \). HINT:

\[
A = \begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.
\]

Since \( H \) is in reduced row echelon form and \( H \) contains 3 pivots, then \( \text{rank}(A) = 3 \). A basis for the row space is given by the nonzero rows of \( H \): \([1, 0, -1, 0], [0, 1, 1, 0], [0, 0, 0, 1] \). A basis for the column space of \( A \) is given by the columns of \( A \) which correspond to pivot columns of \( H \):

\[
\begin{bmatrix} 1 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}
\]

3. Assume that \( T \) is a linear transformation where \( T([-1, 1]) = [2, 1, 4] \) and \( T([1, 1]) = [-6, 3, 2] \). Find the standard matrix representation \( A_T \) of \( T \) and a (row) formula for \( T([x, y]) \). HINT:

\[
-\frac{1}{2}[-1, 1] + \frac{1}{2}[1, 1] = [1, 0] \text{ and } \frac{1}{2}[-1, 1] + \frac{1}{2}[1, 1] = [0, 1].
\]

We have

\[
T([1, 0]) = T(\frac{1}{2} [-1, 1] + \frac{1}{2} [1, 1]) = \frac{1}{2} T([-1, 1]) + \frac{1}{2} T([1, 1]) = \frac{1}{2} [2, 1, 4] + \frac{1}{2} [-6, 3, 2] = [-4, 1, 1]
\]

\[
T([-1, 0]) = T(\frac{1}{2} [-1, 1] + \frac{1}{2} [1, 1]) = \frac{1}{2} T([-1, 1]) + \frac{1}{2} T([1, 1]) = \frac{1}{2} [2, 1, 4] + \frac{1}{2} [-6, 3, 2] = [2, 2, 3].
\]

So the columns of \( A_T \) are \( T([1, 0]) = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix} \) and \( T([-1, 1])^T = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \).

\[
A_T = \begin{bmatrix} -4 & -2 \\ 1 & 2 \\ -1 & 3 \end{bmatrix}
\]
4. Give parametric equations for the line in $\mathbb{R}^3$ through the point $(-1, 3, 0)$ with direction vector $\vec{d} = [-2, -1, 4]$.

We use the translation vector from $(0, 0, 0)$ to $(-1, 3, 0)$, so $\vec{a} = [-1 - 0, 3 - 0, 0 - 0] = [-1, 3, 0]$. Then the line is given parametrically as $\vec{x} = t \vec{d} + \vec{a}$, or

$\begin{align*}
x_1 &= t d_1 + a_1 = -2t - 1 \\
x_2 &= t d_2 + a_2 = -t + 3 \\
x_3 &= t d_3 + a_3 = 4t
\end{align*}$

5. State the definition of a vector space $V$. Include all parts of the definition.

A vector space is a set $V$ of vectors along with an operation of addition $+$ of vectors and multiplication of a vector by a scalar (real number), which satisfy the following. For all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all $r, s \in \mathbb{R}$:

A1. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
A2. $\vec{u} + \vec{w} = \vec{w} + \vec{u}$
A3. There exists $\vec{0} \in V$ such that $\vec{0} + \vec{v} = \vec{v}$
A4. $\vec{0} + (-\vec{v}) = \vec{0}$
S1. $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$
S2. $(r + s)\vec{v} = r\vec{v} + s\vec{v}$
S3. $r(s\vec{v}) = (rs)\vec{v}$
S4. $1\vec{v} = \vec{v}$. 
6. Is the set of all polynomials of degree greater than 3 together with the zero polynomial a subspace of the vector space of all polynomials, \( P \)? Explain your answer.

To have a subspace, we need closure under vector addition and scalar multiplication. But for \( p(x) = 3x^5 - 2x^4 + x^2 \)
and \( q(x) = -3x^5 + 2x^4 \), polynomials of degree greater than 3, we have
\[
\begin{align*}
p(x) + q(x) &= (3x^5 - 2x^4 + x^2) + (-3x^5 + 2x^4) = x^2,
\end{align*}
\]
which is not a polynomial of degree greater than 3. So the set is not closed under vector addition and

\[ \text{NOT a subspace of } P. \]

7. Find the coordinate vector of the polynomial \( p(x) = 4x^3 - 9x^2 + x \) relative to the ordered basis

\[
B' = \{(x - 1)^3, (x - 1)^2, (x - 1), 1\}
\]

of the vector space \( P_3 \) of polynomials of degree 3 or less. Use ordered bases in your answer. HINT:

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 4 \\
-3 & 1 & 0 & 0 & -9 \\
3 & -2 & 1 & 0 & 1 \\
-1 & 1 & -1 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & -5 \\
0 & 0 & 0 & 1 & -4
\end{bmatrix} = H.
\]

Introduce the ordered basis \( B = (x^3, x^2, x, 1) \) and express all vectors as coordinate vectors relative to \( B \):

\[
(x - 2)^3_B = (x^3 - 3x^2 + 3x - 1)_B = [1, -3, 3, -1],
\]

\[
(x - 2)^2_B = (x^2 - 2x + 1)_B = [0, 1, -2, 1],
\]

\[
(x - 2)_B = [0, 0, 1, -1], \quad 1_B = [0, 0, 0, 1],
\]

and

\[
p(x)_B = (4x^3 - 9x^2 + x)_B = [4, -9, 1, 0].
\]

To find \( p(x)_B \), we need to express \( p(x)_B \) as a linear combination of the coordinate vectors relative to \( B \) of the elements of \( B' \).

This leads to a system of equations with augmented matrix \( A \) given above. From \( H \), we see that the coefficients are 4, 3, -5, -4 so that

\[
p(x)_B = [4, 3, -5, -4].
\]
8. (a) Let \( V \) and \( V' \) be vector spaces. State the definition of "\( T \) is a linear transformation that maps vector space \( V \) into vector space \( V' \)."

A transformation \( T : V \rightarrow V' \) is a linear transformation if for all \( \mathbf{u}, \mathbf{v} \in V \) and all \( r \in \mathbb{R} \) we have

1. \( T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \),
2. \( T(r\mathbf{v}) = rT(\mathbf{v}) \).

(b) Let \( F \) be the vector space of all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) (see Example 3.1.3), and let \( D \) be its subspace of all differentiable functions. Show that differentiation is a linear transformation of \( D \) into \( F \).

Let \( T : D \rightarrow F \) be defined as \( T(f) = f' \). Let \( f, g \in D \) and let \( r \in \mathbb{R} \). Since the derivative of a sum is the sum of derivatives, then

\[ T(f + g) = (f + g)' = f' + g' = T(f) + T(g). \]

Since the derivative of a multiple of a function is the multiple times the derivative, then

\[ T(rf) = (rf)' = rf' = rT(f). \]

Therefore, \( T \) is linear.

**Bonus.** Solve the system of equations and express the solution set as a \( k \)-flat for:

\[
\begin{align*}
x_1 + 4x_2 - 2x_3 &= 4 \\
2x_1 + 7x_2 - x_3 &= -2 \\
x_1 + 3x_2 + x_3 &= -6
\end{align*}
\]

(12 points) HINT:

\[
A = \begin{bmatrix} 1 & 4 & -2 & 4 \\ 2 & 7 & -1 & -2 \\ 1 & 3 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 10 & -36 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.
\]

Since \( A \) is the augmented matrix for the system of equations, we have

from \( H \) that \( x_1 + 10x_3 = -36 \) or \( x_1 = -36 - 10x_3 \) or with \( t = x_3 \) we have

\[
\begin{align*}
x_1 &= -36 - 10t \\
x_2 &= 10 + 3x_3 \\
x_3 &= t
\end{align*}
\]

We take \( \mathbf{a} = \begin{bmatrix} -36 \\ 10 \\ 0 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix} \),

so the \( k \)-flat in \( \mathbf{a} + \mathbf{w} = \begin{bmatrix} -36 \\ 10 \\ 0 \end{bmatrix} + \mathbf{y} \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix} \).