Show all work. Be clear and convince me that you understand what is going on. Each time you perform an elementary row operation, indicate what it is (as in class). Put your final answer in the box, when provided. Each numbered problem is worth 12 points. No calculators and put away your cell phone! As in class, an arrow "$\rightarrow$" indicates row equivalence.

1. Find a basis for $\text{span}([1, 2, 1, 2], [2, 1, 0, -1], [-1, 4, 3, 8], [0, 3, 2, 5])$. Explain your reasoning. HINT:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 4 & 3 & 8 \\ 0 & 3 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/3 & -4/3 \\ 0 & 1 & 2/3 & 5/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H$$

$$B = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 4 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & -1 & 8 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = J.$$ Matrix $B$ in its column space gives the given span. To find a basis for the column space, we row reduce $B$ to get $J$ and then take the columns of $B$ which correspond to pivot containing columns of $J$. A basis for the column space of $B$ is given by the first two columns of $J$. That is, a basis for the given space is

$$\begin{bmatrix} 1, 2, 1, 2 \\ 2, 1, 0, -1 \end{bmatrix}$$

4. Consider the linear transformation $T([x_1, x_2]) = [2x_1 - x_2, x_1 + x_2, x_1 + 3x_2]$ from $\mathbb{R}^2$ into $\mathbb{R}^3$. Give the standard matrix representation for $T$. Explain your answer.

Apply $T$ to the standard basis vectors for $\mathbb{R}^2$:

$$T(\mathbf{i}) = T([1, 0]) = [2, 1, 1]$$

$$T(\mathbf{j}) = T([0, 1]) = [-1, 1, 3].$$

The desired matrix is

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}.$$
2. Find a basis for the row space and the column space of the matrix:

\[
A = \begin{bmatrix}
0 & 2 & 3 & 1 \\
-4 & 4 & 1 & 4 \\
3 & 3 & 2 & 0 \\
-4 & 0 & 1 & 2
\end{bmatrix}
\]

HINT:

\[
A = \begin{bmatrix}
0 & 2 & 3 & 1 \\
-4 & 4 & 1 & 4 \\
3 & 3 & 2 & 0 \\
-4 & 0 & 1 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & -1/2 \\
0 & 1 & 0 & 1/2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = H \text{ and } B = \begin{bmatrix}
0 & -4 & 3 & -4 \\
2 & 4 & 3 & 0 \\
3 & 1 & 2 & 1 \\
1 & 4 & 0 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 22/25 \\
0 & 1 & 0 & 7/25 \\
0 & 0 & 1 & -24/25 \\
0 & 0 & 0 & 0
\end{bmatrix} = J.
\]

A basis for the row space of \(A\) is given by the nonzero rows of \(H\).

A basis for the column space of \(A\) is given by the columns of \(J\) which correspond to pivot columns of \(H\). Thus a basis for the column space of \(A\) is:

\[
\begin{bmatrix}
0 \\
-9 \\
3 \\
-4
\end{bmatrix}, \begin{bmatrix}
2 \\
4 \\
3 \\
1
\end{bmatrix}, \begin{bmatrix}
3 \\
1 \\
2 \\
1
\end{bmatrix}
\]

3. Find a basis for the nullspace of \(A = \begin{bmatrix}
1 & 3 & 0 & -1 & 2 \\
0 & -2 & 4 & -2 & 0 \\
3 & 11 & -4 & -1 & 6 \\
2 & 5 & 3 & -4 & 0
\end{bmatrix}\), HINT:

\[
A = \begin{bmatrix}
1 & 3 & 0 & -1 & 2 \\
0 & -2 & 4 & -2 & 0 \\
3 & 11 & -4 & -1 & 6 \\
2 & 5 & 3 & -4 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 2 & 26 \\
0 & 1 & 0 & -1 & -8 \\
0 & 0 & 1 & -1 & -4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} = H.
\]

Explain your reasoning.

The nullspace of \(A\) is the set of all \(x\) which are solutions to \(Ax = 0\). It can be the system associated with \(H\):

\[
\begin{align*}
x_1 + 2x_4 + 26x_5 &= 0 \\
x_2 - x_4 - 8x_5 &= 0 \\
x_3 + x_4 - 4x_5 &= 0 \\
x_4 &= x_4 \\
x_5 &= x_5
\end{align*}
\]

\[
\begin{align*}
x_1 &= -2x_4 - 26x_5 \\
x_2 &= x_4 \\
x_3 &= x_4 + 4x_5 \\
x_4 &= x_4 \\
x_5 &= 25
\end{align*}
\]

\[
\begin{bmatrix}
-2 \\
1 \\
1 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-26 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
5. Show that the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, x_1 + x_2, x_1]$ is invertible, and find a formula for its inverse, $T^{-1}([x_1, x_2, x_3])$. Explain your reasoning. HINT:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = H.$$ Notice that $T([1]) = [1, 1, 1]$, $T([1]) = [1, 1, 0]$, $T([0]) = [1, 0, 0]$. The matrix representing $T$ in $[T([1]) \ T([1]) \ T([0])] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. From this information, the inverse of this matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$. Now $T^{-1}(z) = A^{-1}z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

6. Find the vector equation for the plane in $\mathbb{R}^4$ that passes through the points $(1, 2, 1, 3)$, $(4, 1, 2, 1)$, and $(3, 1, 2, 0)$.

Vectors in the plane are (say)

$\vec{d}_1 = [1-1, 1-2, 2-1, 1-3] = [3, -1, -1, -2]$ and $\vec{d}_2 = [3-1, 1-2, 2-1, 0-3] = [2, -1, 1, -3]$.

A translation vector is a vector from $(0, 0, 0, 0)$ to $(1, 2, 1, 3)$; say $\vec{a} = [1, 2, 1, 3]$.

The plane is all $\vec{x} = r\vec{d}_1 + s\vec{d}_2 + \vec{a}$ where $r, s \in \mathbb{R}$ and

$\vec{d}_1 = [3, -1, -1, -2]$, $\vec{d}_2 = [2, -1, 1, -3]$, and $\vec{a} = [1, 2, 1, 3]$.
7. Solve the system of linear equations

\[
\begin{align*}
    x_1 + 4x_2 - 2x_3 &= 4 \\
    2x_1 + 7x_2 - x_3 &= -2 \\
    x_1 + 3x_2 + x_3 &= -6
\end{align*}
\]

and write the solution set parametrically as a k-flat (that is, as a translation of a vector space). HINT:

\[
A = \begin{bmatrix}
    1 & 4 & -2 & 4 \\
    2 & 7 & -1 & -2 \\
    1 & 3 & 1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
    1 & 0 & 10 & -36 \\
    0 & 1 & -3 & 10 \\
    0 & 0 & 0 & 0
\end{bmatrix} = H.
\]

The system of equations associated with \( H \) is

\[
\begin{align*}
    -x_1 + 10x_3 &= -36 \\
    7x_2 - 3x_3 &= 10 \\
    0 &= 0
\end{align*}
\]

with \( t = x_3 \) as a free variable (or parameter) we have parametrically:

\[
\begin{align*}
    x_1 &= -10t - 36 \\
    x_2 &= 3t + 10 \\
    x_3 &= t
\end{align*}
\]

8. State the definition of vector space.

A vector space is a set \( V \) of vectors along with an operation of addition \( + \) of vectors and multiplication of a vector by a scalar such that for all \( \vec{u}, \vec{v} \in V \) and for all scalars \( r \) and \( s \):

\[
\begin{align*}
    (A1) \quad (\vec{u} + \vec{v}) + \vec{w} &= \vec{u} + (\vec{v} + \vec{w}) \\
    (A2) \quad \vec{u} + \vec{v} &= \vec{v} + \vec{u} \\
    (A3) \quad \text{there exists } \vec{0} \text{ in } V \text{ such that } \vec{0} + \vec{u} = \vec{u} \\
    (A4) \quad \vec{u} + (-\vec{v}) &= \vec{0}
\end{align*}
\]

\[
\begin{align*}
    (S1) \quad r(\vec{u} + \vec{v}) &= r\vec{u} + r\vec{v} \\
    (S2) \quad (r + s)\vec{u} &= r\vec{u} + s\vec{u} \\
    (S3) \quad r(s\vec{u}) &= (rs)\vec{u} \\
    (S4) \quad 1\vec{u} = \vec{u}
\end{align*}
\]
Bonus. Do one of the following. (12 points)

(a) A linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ preserves length if $\|T(\vec{u})\| = \|\vec{u}\|$ for all $\vec{u} \in \mathbb{R}^2$. It preserves angle if the angle between $T(\vec{u})$ and $T(\vec{v})$ is the same as the angle between $\vec{u}$ and $\vec{v}$ for all $\vec{u}, \vec{v} \in \mathbb{R}^2$. It preserves the dot product if $T(\vec{u}) \cdot T(\vec{v}) = \vec{u} \cdot \vec{v}$ for all $\vec{u}, \vec{v} \in \mathbb{R}^2$. Use the familiar equation that describes the dot product $\vec{u} \cdot \vec{v}$ geometrically to prove that if a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ preserves both length and angle, then it also preserves the dot product.

(b) Use the parts of the definition of a vector space to prove that for any $\vec{u}, \vec{v}, \vec{w}$ in a vector space, we have that $\vec{u} + \vec{v} = \vec{u} + \vec{w}$ implies that $\vec{u} = \vec{w}$. Justify each step in your proof.

(a) Recall that $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta_1$ where $\theta_1$ is the angle between $\vec{u}$ and $\vec{v}$. Show similarly, $T(\vec{u}) \cdot T(\vec{v}) = \|T(\vec{u})\| \|T(\vec{v})\| \cos \theta_2$ where $\theta_2$ is the angle between $T(\vec{u})$ and $T(\vec{v})$. As $T$ preserves both length and angle then $\|\vec{u}\| = \|T(\vec{u})\|$, $\|\vec{v}\| = \|T(\vec{v})\|$, and $\theta_1 = \theta_2$. So $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta_1 = \|T(\vec{u})\| \|T(\vec{v})\| \cos \theta_2 = T(\vec{u}) \cdot T(\vec{v})$. Q.E.D.

(b) Suppose $\vec{u} + \vec{v} = \vec{u} + \vec{w}$. Then we add $-\vec{u}$ to both sides of the equation and we get:

1. $(\vec{u} + \vec{v}) + (-\vec{u}) = (\vec{u} + \vec{w}) + (-\vec{u})$
2. $(\vec{v} + \vec{w}) + (-\vec{u}) = (\vec{w} + \vec{u}) + (-\vec{u})$ by commutativity, A2
3. $\vec{v} + (\vec{u} - \vec{u}) = \vec{w} + (\vec{u} - \vec{u})$ by associativity, A1
4. $\vec{v} + 0 = \vec{w} + 0$ by additive inverse, A4
5. $\vec{v} = \vec{w}$ by additive identity, A3.

Q.E.D.