Chapter 1. Vectors, Matrices, and Linear Spaces

1.1. Vectors in Euclidean Spaces

Note. In the first half of this section of the text, Fraleigh and Beauregard motivate the study and properties of vectors with the physical model of forces.

Note. Euclidean 1-space is simply the line, denoted $\mathbb{R}$. Euclidean 2-space is the collection of all ordered pairs (x, y) where $x, y \in \mathbb{R}$, and is denoted $\mathbb{R}^2$; you are familiar with $\mathbb{R}^2$ since you have used the Cartesian plane in algebra and calculus. We can generalize these ideas as follows.

Definition 1.A. The space $\mathbb{R}^n$, or *Euclidean n-space*, is either **(1)** the collection of all n-tuples of the form $(x_1, x_2, \ldots, x_n)$ where the x_i's are real numbers (the n-tuples are called *points*), or **(2)** the collection of all n-tuples of the form $[x_1, x_2, \ldots, x_n]$ where the x_i's are real numbers (the n-tuples are called *vectors*).

Note. $\mathbb{R}^1$ is just the collection of real numbers (which we know to have an algebraic structure—addition and subtraction, say). $\mathbb{R}^2$ is the collection of all points in the Cartesian plane.

Note. We, *as yet*, make no distinction between points and vectors.

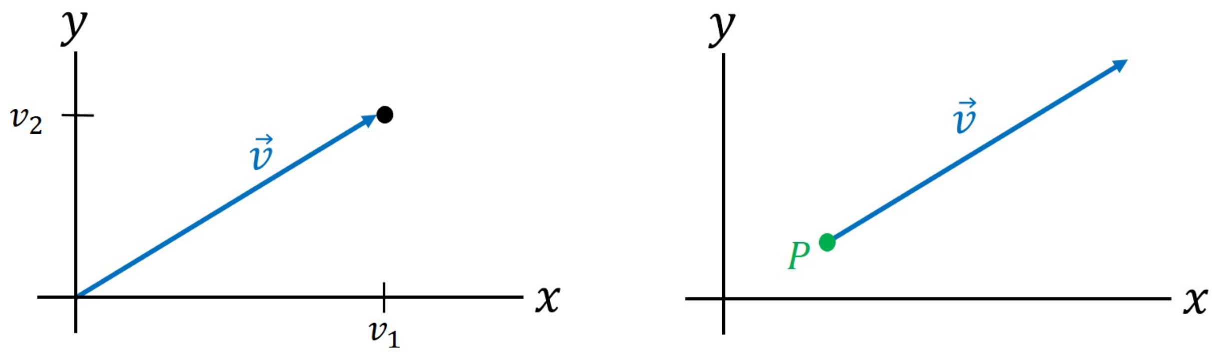
Note. As is common, the text uses bold faced lower case letters to denote vectors: $\mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{w}$. In these class notes and when writing by hand we denote vectors with lower case letters with little arrows over them: $\vec{x}, \vec{y}, \vec{v}, \vec{w}$. We will be dealing with points, vectors, and numbers (which we will call "scalars") so it is essential that we use a notation when distinguishes these different mathematical objects from one another.

Definition. For $\vec{x} \in \mathbb{R}^n$, say $\vec{x} = [x_1, x_2, \ldots, x_n]$, the *ith component* of $\vec{x}$ is x_i.

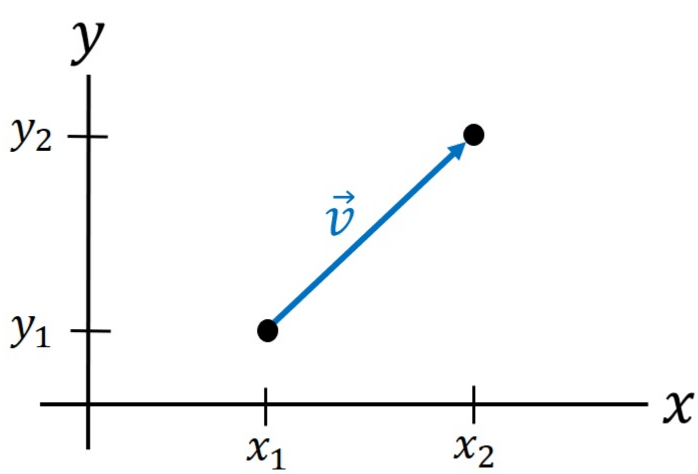
Definition. Two vectors in $\mathbb{R}^n$, $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ are *equal* if each of their components are equal. The *zero vector*, $\vec{0}$, in $\mathbb{R}^n$ is the vector of all zero components.

Note. You are probably familiar with the idea from physics of describing forces and velocities as "vector quantities" with both magnitude and direction. We use this as motivation for our approach to vectors (in particular when defining vector sums).

Note. We have the following **geometric interpretation** of vectors: A vector $\vec{v} \in \mathbb{R}^2$ can be drawn in *standard position* in the Cartesian plane by drawing an arrow from the **point** $(0, 0)$ to the **point** (v_1, v_2) where $\vec{v} = [v_1, v_2]$:



On the right of this picture, $\vec{v}$ is *translated* to point P. Notice that both of these are representations of the same vector $\vec{v}$. The vector in $\mathbb{R}^2$ with its tail at point (x_1, y_1) and its head at point (x_2, y_2) is $\vec{v} = [x_2 - x_1, y_2 - y_1]$:



These ideas can each be extended to vectors in $\mathbb{R}^n$ in the obvious way.

Note. In physics, forces are represented by "arrows" (or *vectors*) and if two forces $\vec{F}_1$ and $\vec{F}_2$ are applied to an object, the resulting force $\vec{F}_1 + \vec{F}_2$ satisfies a "parallelogram" property:

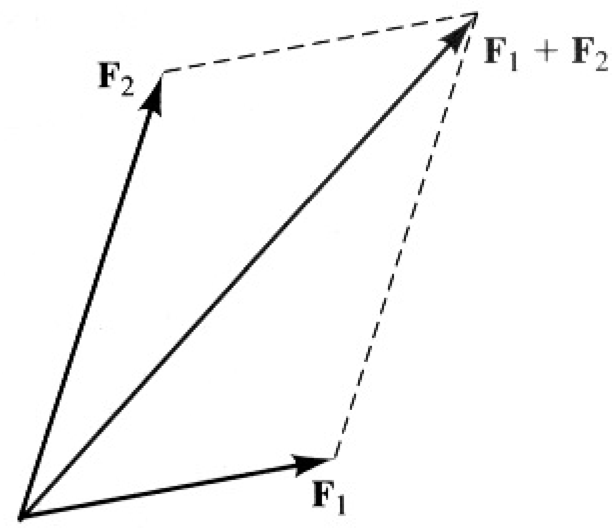
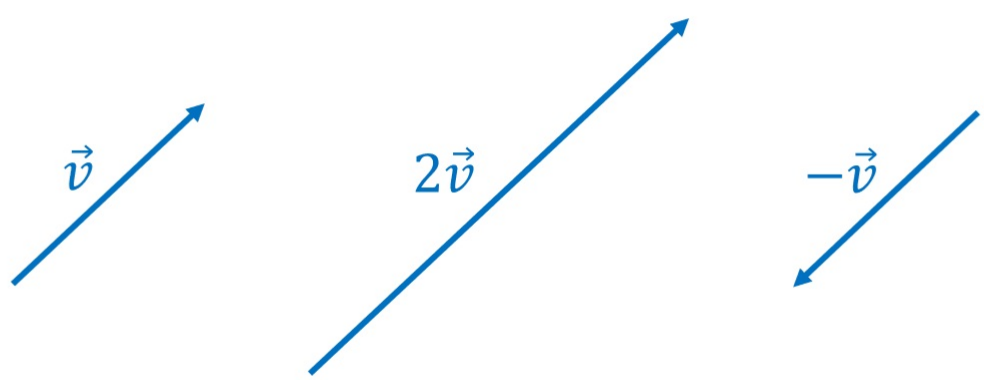


Figure 1.1.5, page 5

You can also talk about scaling a force by a constant c (we call these constants *scalars* — as opposed to vectors and points):



With physics as our motivation, we now define properties of addition and scalar multiplication of vectors.

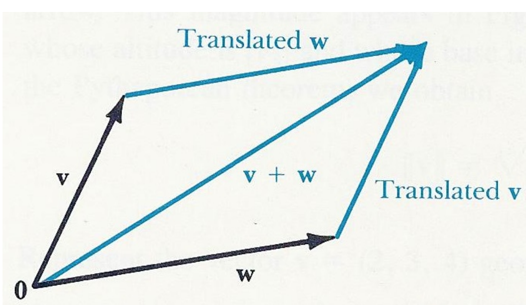
Definition 1.1. Let $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ be vectors in $\mathbb{R}^n$ and let $r \in \mathbb{R}$ be a scalar. Define

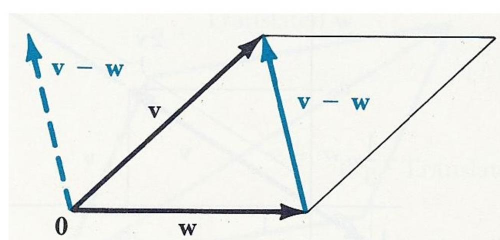
1. *Vector addition*: $\vec{v} + \vec{w} = [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n]$,

2. *Vector subtraction*: $\vec{v} - \vec{w} = [v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n]$, and

3. *Scalar multiplication*: $r\vec{v} = [rv_1, rv_2, \ldots, rv_n]$.

Note. By Definition 1.1, we can associate the parallelogram property with the addition and subtraction of vectors in $\mathbb{R}^n$ (this claim is algebraically established in Example 1.2.7). So to compute the vector sum $\vec{v} + \vec{w}$ geometrically, we can place $\vec{v}$ in standard position and translate $\vec{w}$ so that its tail coincides with the head of $\vec{v}$. Then the point at the head of this translation of $\vec{w}$ determines the head of $\vec{v} + \vec{w}$ when in standard position (see Figure 1.6 below). We can similarly interpret $\vec{v} - \vec{w}$ as a vector with its head at the head of $\vec{v}$ and its tail at the head of $\vec{w}$, where $\vec{v}$ and $\vec{w}$ are in standard position (see Figure 1.8 below).





Examples. Page 16 numbers 10 and 14.

Theorem 1.1. Properties of Vector Algebra in $\mathbb{R}^n$.

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and let r, s be scalars in $\mathbb{R}$. Then

A1. Associativity of Vector Addition. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

A2. Commutivity of Vector Addition. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

A3. Additive Identity. $\vec{0} + \vec{v} = \vec{v}$

A4. Additive Inverses. $\vec{v} + (-\vec{v}) = \vec{0}$

S1. Distribution of Scalar Multiplication over Vector Addition.

$$r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$$

S2. Distribution of Scalar Addition over Scalar Multiplication.

$$(r + s)\vec{v} = r\vec{v} + s\vec{v}$$

S3. Associativity. $r(s\vec{v}) = (rs)\vec{v}$

S4. "Preservation of Scale." $1\vec{v} = \vec{v}$

Note. The proofs of A2 and S2 are given in Examples 1.1.3 and 1.1.4 of the text.

Examples. Page 17 Number 40a (prove A1) and Page 17 Number 41(a) (prove S1).

Definition 1.2. Two nonzero vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are *parallel*, denoted $\vec{v} \parallel \vec{w}$, if one is a scalar multiple of the other. If $\vec{v} = r\vec{w}$ with $r > 0$, then $\vec{v}$ and $\vec{w}$ have the *same direction* and if $\vec{v} = r\vec{w}$ with $r < 0$ then $\vec{v}$ and $\vec{w}$ have *opposite directions.*

Example. Page 16 number 22.

Definition 1.3. Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$ and scalars $r_1, r_2, \ldots, r_k \in \mathbb{R}$, the vector

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k = \sum_{\ell=1}^{k} r_\ell \vec{v}_\ell$$

is a *linear combination* of the given vectors with the given scalars as *scalar coefficients.*

Note. Sometimes there are "special" vectors for which it is easy to express a vector in terms of a linear combination of these special vectors.

Definition. The *standard basis vectors in* $\mathbb{R}^2$ are $\hat{\imath} = [1, 0]$ and $\hat{\jmath} = [0, 1]$. The *standard basis vectors in* $\mathbb{R}^3$ are

$$\hat{\imath} = [1, 0, 0], \hat{\jmath} = [0, 1, 0], \text{ and } \hat{k} = [0, 0, 1].$$

Note. It's easy to write a vector in terms of the standard basis vectors:

$$\vec{b} = [b_1, b_2] = b_1[1, 0] + b_2[0, 1] = b_1\hat{\imath} + b_2\hat{\jmath} \text{ and}$$

$$\vec{b} = [b_1, b_2, b_3] = b_1[1, 0, 0] + b_2[0, 1, 0] + b_3[0, 0, 1] = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}.$$

Definition. In $\mathbb{R}^n$, the *rth standard basis vector*, denoted $\hat{e}_r$, is

$$\hat{e}_r = [0, 0, \ldots, 0, 1, 0, \ldots, 0],$$

where the rth component is 1 and all other components are 0.

Notice. A vector $\vec{b} \in \mathbb{R}^n$ can be **uniquely** expressed in terms of the standard basis vectors:

$$\vec{b} = [b_1, b_2, \ldots, b_n] = b_1\hat{e}_1 + b_2\hat{e}_2 + \cdots + b_n\hat{e}_n = \sum_{\ell=1}^{n} b_\ell \hat{e}_\ell.$$

Definition. If $\vec{v} \in \mathbb{R}^n$ is a nonzero vector, then the *line along* $\vec{v}$ is the collection of all vectors of the form $r\vec{v}$ for some scalar $r \in \mathbb{R}$ (notice $\vec{0}$ is on all such lines). For two nonzero nonparallel vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$, the collection of all possible linear combinations of these vectors: $r\vec{v} + s\vec{w}$ where $r, s \in \mathbb{R}$, is the *plane spanned by* $\vec{v}$ *and* $\vec{w}$.

Definition. A *column vector* in $\mathbb{R}^n$ is a representation of a vector as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

A *row vector* in $\mathbb{R}^n$ is a representation of a vector as

$$\vec{x} = [x_1, x_2, \ldots, x_n].$$

The *transpose* of a row vector, denoted $\vec{x}^T$, is a column vector, and conversely:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T = [x_1, x_2, \ldots, x_n], \text{ and } [x_1, x_2, \ldots, x_n]^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Note. A linear combination of column vectors can easily be translated into a system of linear equations:

$$r \begin{bmatrix} 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 19 \end{bmatrix} \Longleftrightarrow \begin{array}{rcr} r - 2s & = & -1 \\ 3r + 5s & = & 19 \end{array}.$$

Note. Throughout what follows, we will use set notation. Informally, a *set* is a collection of objects called *elements*. Most often, the sets with which we deal will be sets of vectors. We might have a set $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, in which case we write, for example, $\vec{v}_1 \in V$ to represent "$\vec{v}_1$ is an element of V." We may describe a set by giving properties of the elements of the set. For example, $V = \{\vec{v} = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_3 = 0\}$ is the set of all vectors $\vec{v} = [x_1, x_2, x_3]$ in $\mathbb{R}^3$ such that (the symbol "$|$" should be read "such that") the third component is 0, $x_3 = 0$. We should note that elements are either in a set or not in the set; elements are not in a set multiple times (they are not repeated in the set). For a more formal treatment of sets you might start with my online notes on Introduction to Set Theory.

Definition 1.4. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$. The *span* of these vectors is the **set** of all linear combinations of them, denoted $\text{sp}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$:

$$\begin{aligned} \text{sp}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) &= \{r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k \mid r_1, r_2, \ldots, r_k \in \mathbb{R}\} \\ &= \left\{ \sum_{\ell=1}^{k} r_\ell \vec{v}_\ell \,\middle|\, r_1, r_2, \ldots, r_k \in \mathbb{R} \right\}. \end{aligned}$$

Example. Page 16 number 28.