Chapter 1. Vectors, Matrices, and Linear Spaces

1.3. Matrices and Their Algebra

Note. We define a "matrix" and give a way to add and multiply matrices. We state and prove some properties of this addition and multiplication (that is, this "algebra").

Definition. A *matrix* is a rectangular array of numbers. An $m \times n$ matrix is a matrix with m *rows* and n *columns*:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Definition 1.8. Let $A = [a_{ik}]$ be an $m \times n$ matrix and let $B = [b_{kj}]$ be an $n \times s$ matrix. The *matrix product* AB is the $m \times s$ matrix $C = [c_{ij}]$ where c_{ij} is the dot product of the ith row vector of A and the jth column vector of B: $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Note. We can draw a picture of this process as:

$$AB = [c_{ij}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1s} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{ns} \end{bmatrix}$$

Note 1.3.A. For $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ we have

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

So any product of the form $A\vec{x}$ is a linear combination of the columns of matrix A with coefficients as the components of vector $\vec{x}$.

Example. Page 46 Number 16.

Definition. The *main diagonal* of an $n \times n$ matrix is the set $\{a_{11}, a_{22}, \ldots, a_{nn}\}$. A square matrix which has zeros off the main diagonal is a *diagonal matrix*. We denote the $n \times n$ diagonal matrix with all diagonal entries 1 as $\mathcal{I}$:

$$\mathcal{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Definition 1.9/1.10. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The *sum* $A + B$ is the $m \times n$ matrix $C = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$. Let r be a scalar. Then rA is the matrix $D = [d_{ij}]$ where $d_{ij} = ra_{ij}$.

Example. Page 46 Number 6.

Definition 1.11. Matrix B is the *transpose* of A, denoted $B = A^T$, if $b_{ij} = a_{ji}$. If A is a matrix such that $A = A^T$ then A is *symmetric*.

Example. Page 47 Number 38. If A is square, then $A + A^T$ is symmetric.

Proof. Let $A = [a_{ij}]$ then $A^T = [a_{ji}]$. Let $C = [c_{ij}] = A + A^T = [a_{ij}] + [a_{ji}] = [a_{ij} + a_{ji}]$. Notice $c_{ij} = a_{ij} + a_{ji}$ and $c_{ji} = a_{ji} + a_{ij}$, therefore $C = A + A^T$ is symmetric. ∎

Theorem 1.3.A. Properties of Matrix Algebra.

Let A, B, and C be matrices such that the sums and products below are defined and let r and s be scalars. Then

1. Commutative Law of Addition: $A + B = B + A$

2. Associative Law of Addition: $(A + B) + C = A + (B + C)$

3. Additive Identity: $A + 0 = 0 + A = A$ (here "0" represents the $m \times n$ matrix of all zeros)

4. Left Distribution Law: $r(A + B) = rA + rB$

5. Right Distribution Law: $(r + s)A = rA + sA$

6. Associative Law of Scalar Multiplication: $(rs)A = r(sA)$

7. Scalars "Pull Through": $(rA)B = A(rB) = r(AB)$

8. Associativity of Matrix Multiplication: $A(BC) = (AB)C$

9. Matrix Multiplicative Identity: $\mathcal{I}A = A = A\mathcal{I}$

10. Distributive Laws of Matrix Multiplication: $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.

Note. The proof of the Left Distributive Law, $A(B + C) = AB + AC$, is given in Example 11 on page 45.

Page 47 Number 33. Let A, B, and C be matrices where the products $(AB)C$ and $A(BC)$ are defined. Then matrix multiplication is associative: $(AB)C = A(BC)$.

Example 1.3.A. Show that $\mathcal{I}A = A\mathcal{I} = A$ for $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $\mathcal{I}$ is 3×3.

Note 1.3.B. Properties of the Transpose Operator.

$$(A^T)^T = A \qquad (A + B)^T = A^T + B^T \qquad (AB)^T = B^T A^T.$$

Example. Page 47 number 32. Prove $(AB)^T = B^T A^T$.

Proof. Let $C = [c_{ij}] = (AB)^T$. The (i, j)-entry of AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$, so $c_{ij} = \sum_{k=1}^{n} a_{jk}b_{ki}$. Let $B^T = [b_{ij}]^T = [b^t_{ij}] = [b_{ji}]$ and $A^T = [a_{ij}]^T = [a^t_{ij}] = [a_{ji}]$. Then the (i, j)-entry of $B^T A^T$ is

$$\sum_{k=1}^{n} b^t_{ik}a^t_{kj} = \sum_{k=1}^{n} b_{ki}a_{jk} = \sum_{k=1}^{n} a_{jk}b_{ki} = c_{ij}$$

and therefore $C = (AB)^T = B^T A^T$. ∎