Chapter 1. Vectors, Matrices, and Linear Spaces

1.4. Solving Systems of Linear Equations

Note. We give an algorithm for solving a system of linear equations (called the Gauss-Jordan method). This algorithm will give the unique solution when it exists, give a way to find multiple solutions when they exist, and reveal when no solution exists.

Definition. A *system of m linear equations* in the n unknowns $x_1, x_2, \ldots, x_n$ is a system of the form:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array}$$

Note. The above system can be written as $A\vec{x} = \vec{b}$ where A is the *coefficient matrix* and $\vec{x}$ is the vector of variables. A *solution* to the system is a vector $\vec{s}$ such that $A\vec{s} = \vec{b}$.

Defninition. The *augmented matrix* for the above system is

$$[A \mid \vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

Note 1.4.A. We will perform certain operations on the augmented matrix which correspond to the following manipulations of the system of equations:

1. interchange two equations,

2. multiply an equation by a nonzero constant,

3. replace an equation by the sum of itself and a multiple of another equation.

Definition. The following are *elementary row operations*:

1. interchange row i and row j (denoted $R_i \leftrightarrow R_j$),

2. multiplying the ith row by a nonzero scalar s (denoted $R_i \to sR_i$), and

3. adding the ith row to s times the jth row (denoted $R_i \to R_i + sR_j$).

If matrix A can be obtained from matrix B by a series of elementary row operations, then A is *row equivalent* to B, denoted $A \sim B$ (or sometimes denoted $A \to B$).

Notice. These operations correspond to the above manipulations of the equations and so:

Theorem 1.6. Invariance of Solution Sets Under Row Equivalence.
If $[A \mid \vec{b}] \sim [H \mid \vec{c}]$ then the linear systems $A\vec{x} = \vec{b}$ and $H\vec{x} = \vec{c}$ have the same solution sets.

Definition 1.12. A matrix is in *row-echelon form* (abbreviated "REF") if

1. all rows containing only zeros appear below rows with nonzero entries, and

2. the first nonzero entry in any row appears in a column to the right of the first nonzero entry in any preceding row.

For such a matrix, the first nonzero entry in a row is the *pivot* for that row.

Example. Which of the following is in row echelon form?

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 6 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Note. If an augmented matrix is in row-echelon form, we can use the method of *back substitution* to find solutions.

Example 1.4.A. Solve the system

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & x_3 & = & 4 \\ & & x_2 & - & x_3 & = & -1 \\ & & & & x_3 & = & 3. \end{array}$$

Definition 1.13. A linear system having no solution is *inconsistent*. If it has one or more solutions, it is *consistent*.

Example 1.4.B. Is this system consistent or inconsistent:

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & - & x_2 & + & 3x_3 & = & 1 \\ 3x_1 & & & + & 2x_3 & = & 3? \end{array}$$

Example 1.4.C. Is this system consistent or inconsistent:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & - & x_2 & + & 3x_3 & = & 1 \\ 3x_1 & & & + & 2x_3 & = & 2? \end{array}$$

(HINT: This system has multiple solutions. Express the solutions in terms of an unknown parameter r).

Note. In the above example, r is a "free variable" and the *general solution* is in terms of this free variable.

Note. Reducing a Matrix to Row-Echelon Form.

1. If the first column is all zeros, "mentally cross it off." Repeat this process as necessary.

2(a). Use row interchange if necessary to get a nonzero entry (pivot) p in the top row of the remaining matrix.

2(b). For each row R below the row containing this entry p, add $-r/p$ times the row containing p to R where r is the entry of row R in the column which contains pivot p. (This gives all zero entries below pivot p.)

3. "Mentally cross off" the first row and first column to create a smaller matrix. Repeat the process **(1) - (3)** until either no rows or no columns remain.

Example. Page 68 number 2(a).

Example. Page 69 number 16(a). (Put the associated augmented matrix in row-echelon form and then use substitution.)

Note. The above method is called *Gauss reduction with back substitution.*

Note 1.4.B. The system $A\vec{x} = \vec{b}$ is equivalent to the system (see Note 1.3.A):

$$x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n} = \vec{b}$$

where $\vec{a_i}$ is the ith column vector of A. Therefore, $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in the span of $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}$ (the columns of A).

Definition. A matrix is in *reduced row-echelon form* (abbreviated "RREF") if all the pivots are 1 and all entries above and below pivots are 0.

Examples. Page 68 number 2(b) and Page 69 number 16(b).

Note. The above method is the *Gauss-Jordan method.*

Theorem 1.7. Solutions of $A\vec{x} = \vec{b}$.

Let $A\vec{x} = \vec{b}$ be a linear system and let $[A \mid \vec{b}] \sim [H \mid \vec{c}]$ where H is in row-echelon form.

1. The system $A\vec{x} = \vec{b}$ is inconsistent if and only if $[H \mid \vec{c}]$ has a row with all entries equal to 0 to the left of the partition and a nonzero entry to the right of the partition.

2. If $A\vec{x} = \vec{b}$ is consistent and every column of H contains a pivot, the system has a unique solution.

3. If $A\vec{x} = \vec{b}$ is consistent and some column of H has no pivot, the system has infinitely many solutions, with as many free variables as there are pivot-free columns of H.

Definition 1.14. A matrix that can be obtained from an identity matrix by means of **one** elementary row operation is an *elementary matrix*.

Theorem 1.8. Let A be an $m \times n$ matrix and let E be an $m \times m$ elementary matrix. Multiplication of A **on the left** by E effects the same elementary row operation on A that was performed on the identity matrix to obtain E.

Note. We give the proof now for Row Interchange (Page 71 Number 52) and Row Addition (page 71 Number 54). The proof for Row Scaling is to be given in Exercise 53.

Example 1.4.D. Multiply some 3×3 matrix A by

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to swap Row 1 and Row 2.

Note. If A is row equivalent to B, then we can find C such that $CA = B$ and C is a product of elementary matrices.

Examples. Page 70 Number 44, Page 70 Number 50, Page 71 Number 56.

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