Chapter 1. Vectors, Matrices, and Linear Spaces

1.5. Inverses of Square Matrices

Note. In this section we define the inverse of a matrix and give a technique to find the inverse (when it exists) which uses elementary row operations. As a consequence, elementary matrices play a role in the theory of this section.

Definition 1.15. An $n \times n$ matrix A is *invertible* if there exists an $n \times n$ matrix C such that $AC = CA = \mathcal{I}$. If A is not invertible, it is *singular*.

Note. We'll see examples of matrices that do not have an inverse (in fact, we will classify invertible matrices). When an inverse exists, though, it is unique as we now show.

Theorem 1.9. Uniqueness of an Inverse Matrix.
An invertible matrix has a unique inverse (which we denote A^{-1}).

Proof. Suppose C and D are both inverses of A. Then $(DA)C = \mathcal{I}C = C$ and $D(AC) = D\mathcal{I} = D$. But $(DA)C = D(AC)$ (Theorem 1.3.A(8), Associativity of Matrix Multiplication), so $C = D$. ∎

Example 1.5.A. It is easy to invert an elementary matrix. For example, suppose E_1 interchanges Row 1 and Row 2 of a 3×3 matrix. Suppose E_2 multiplies Row 2 by 7 in a 3×3 matrix. Find the inverses of E_1 and E_2.

Note. The following will be particularly useful when we use elementary matrices to perform row reduction.

Theorem 1.10. Inverses of Products.
Let A and B be invertible $n \times n$ matrices. Then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. By associativity and the assumption that A^{-1} and B^{-1} exist, we have:

$$(AB)(B^{-1}A^{-1}) = [A(BB^{-1})]A^{-1} = (A\mathcal{I})A^{-1} = AA^{-1} = \mathcal{I}.$$

We can similarly show that $(B^{-1}A^{-1})(AB) = \mathcal{I}$. Therefore AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. ∎

Lemma 1.1. Condition for $A\vec{x} = \vec{b}$ to be Solvable for $\vec{b}$.
Let A be an $n \times n$ matrix. The linear system $A\vec{x} = \vec{b}$ has a solution for every choice of column vector $\vec{b} \in \mathbb{R}^n$ if and only if A is row equivalent to the $n \times n$ identity matrix $\mathcal{I}$.

Example. Page 84 Number 12.

Note. The following result shows that in the setting of square matrices, to test if C is an inverse of A we must only check multiplication of A by C on one side.

Theorem 1.11. A Commutivity Property.

Let A and C be $n \times n$ matrices. Then $CA = \mathcal{I}$ if and only if $AC = \mathcal{I}$.

Note 1.5.A. Computation of Inverses.

If $A = [a_{ij}]$, then finding $A^{-1} = [x_{ij}]$ amounts to solving for x_{ij} in:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} = \mathcal{I}.$$

If we treat this as n systems of n equations in n unknowns, then the augmented matrix for these n systems is $[A \mid \mathcal{I}]$. So to compute A^{-1}:

(1) Form $[A \mid \mathcal{I}]$.

(2) Apply Gauss-Jordan method to produce the row equivalent $[\mathcal{I} \mid C]$. If A^{-1} exists, then $A^{-1} = C$.

Note. We now give several conditions which are equivalent to the invertibility of a matrix A. Notice that one of the conditions involves systems of equations.

Example. Page 84 number 4 (also apply this example to a system of equations).

Theorem 1.12. Conditions for A^{-1} to Exist.

The following conditions for an $n \times n$ matrix A are equivalent:

(i) A is invertible.

(ii) A is row equivalent to $\mathcal{I}$.

(iii) $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$ (namely, $\vec{x} = A^{-1}\vec{b}$).

(iv) A can be expressed as a product of elementary matrices.

(v) The span of the column vectors of A is $\mathbb{R}^n$.

Note. In (iv) A is the left-to-right product of the inverses of the elementary matrices corresponding to succesive row operations that reduce A to $\mathcal{I}$.

Example. Page 84 number 2. Express the inverse of $A = \begin{bmatrix} 3 & 6 \\ 3 & 8 \end{bmatrix}$ as a product of elementary matrices.

Solution. We perform the following elementary operations:

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right] \underset{\sim}{\overset{R_2 \to R_2 - R_1}{}} \left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] \underset{\sim}{\overset{R_1 \to R_1 - 3R_2}{}} \left[\begin{array}{cc|cc} 3 & 0 & 4 & -3 \\ 0 & 2 & -1 & 1 \end{array}\right]$$

$$\underset{\sim}{\overset{R_2 \to R_2/2}{}} \left[\begin{array}{cc|cc} 3 & 0 & 4 & -3 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right] \underset{\sim}{\overset{R_1 \to R_1/3}{}} \left[\begin{array}{cc|cc} 1 & 0 & 4/3 & -1 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right].$$

The elementary matrices which accomplish this are:

$$E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_4^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

As in Section 1.3,

$$E_4E_3E_2E_1A = \mathcal{I}$$

and so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}\mathcal{I} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}.$$

Also $\boxed{A^{-1} = E_4E_3E_2E_1.}$ □

Examples. Page 85 Number 24, Page 86 Number 30.

Revised: 9/13/2018